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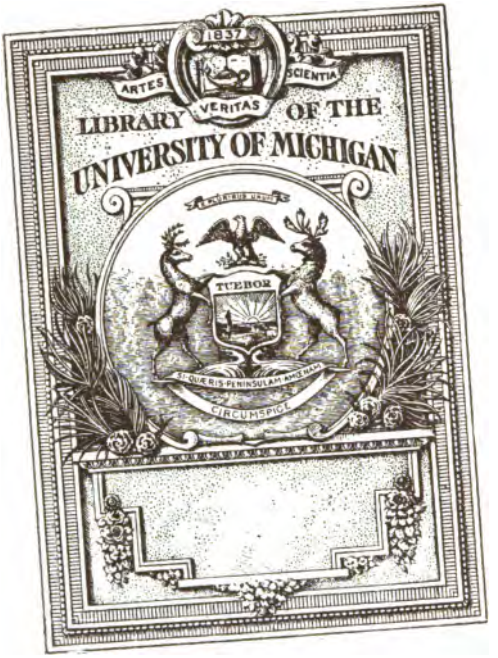

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TREATISE

ON

TRIGONOMETRY.

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Alexander Ziwet 1.8

A TREATISE ON TRIGONOMETRY

BY

W. E. JOHNSON M.A.

FORMERLY SCHOLAR OF KING'S COLLEGE CAMBRIDGE

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Prof. Alexander Ziwet
St.

PREFACE.

THE present work on Trigonometry is intended *first* for students who are beginning the subject but are hoping to proceed to more advanced mathematics, and *secondly* for those who are wishing to revise their study of Trigonometry and to extend it beyond the limits of an ordinary elementary text-book.

It is in Trigonometry that the student is first introduced to almost all the principles which lie at the foundation of mathematics. Here negative and (so-called) imaginary symbols receive a use and meaning; symbols of quantity, of function, and of operation are distinguished from one another; ideas relating to limits, and to the indefinitely small and indefinitely great are brought into prominence; methods of approximation and errors of measurement are recognised; and the principle of continuity is made of service. With the purpose of bringing out clearly the train of reasoning required to establish and expound these principles, it has been thought desirable to make short digressions into Geometry, Algebra, and the Theory of Equations.

A description of the plan of the work may be given. It is divided into two parts, headed, respectively, Geometrical and Algebraical. In the first, Geometrical applications and methods are prominent; in the second, the purely theoretical and analytical side is developed. The first part culminates in the properties of the points and circles connected with triangles and rectilineal figures. A

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chapter has been introduced here on the Geometry of the Triangle which is intended to connect the purely geometrical with the trigonometrical treatment. Here symmetry and completeness demand the introduction of the *cosine-circle* (whose centre, the symmedian point, has the same relation to the centre of gravity that the circumcentre has to the orthocentre). The other related points and circles associated with the names Lemoine, Tucker, Brocard naturally follow. In the treatment of this branch of the subject, the writer is mainly indebted to Prof. Casey's 'Sequel to Euclid': but the theorems and proofs are arranged and formulated anew, and the terminology of 'Modern' and Analytical Geometry is as far as possible avoided. The useful term 'antiparallel' is, however, introduced from the above text-book. The chapter is of course a mere introduction, which it is hoped will be useful for students interested in the modern developments of Geometry.

The algebraical part deals with Logarithms, the application of signs to Trigonometry, developments of the formulæ for the sums of angles, factorisation and summation. All the formulæ are first established in a simple manner independently of De Moivre's Theorem. Several of the methods of proof in Chapter XIV. on Multiple Angles are new. The treatment of infinite series is introduced by a few comprehensive theorems on convergency of series, which will prepare the student for some of the particular difficulties attaching to trigonometrical series. A modification is given of Euler's proof of the Binomial Theorem, by which it is made to depend directly on the Index Theorem. In this way the indeterminateness of fractional powers, which has to be noted in the expansions arising from De Moivre's Theorem, can be treated in close connection with the Binomial Expansion.

The introduction of imaginaries (and of De Moivre's Theorem) is postponed to Chapter XXI., where a careful determination of the values of expansions in the case of fractional indices is given. Here also an attempt is made to justify the use of imaginary indices. In the last chapter an interpretation of $\sqrt{(-1)}$ is given, by which it is hoped that the transition has been made easy from the point of view of Argand to that of Hamilton's quaternions.

To students who are reading the subject for the first time it should be explained that the chapters have been arranged with a regard rather to logical sequence than to gradation in difficulty. Hence a modification in the order of reading the book might be useful to many students. Thus Chapter II. might be postponed until Chapter XIX. is reached: for the proofs in Chapter V. might be substituted those in Chapter XIII., and any part of Part II. might be studied before Chapters IX. and X. are mastered.

The Rev. J. P. Taylor, M.A., gave in the *Quarterly Journal of Mathematics*, Vol. XIII., p. 197, an extremely simple proof that the nine-points circle touches the inscribed and escribed circles. I have ventured to introduce a proof on p. 139 of this theorem, which is in substance borrowed from the above contribution, but has been simplified by the omission of any reference to the method of Inversion.

For much time and trouble spent in correcting the proofs I have to express gratitude to my brother G. W. Johnson, M.A., late scholar of Trinity College, Cambridge, whose careful accuracy has done much to free the work from errors. Corrections of such mistakes as remain and suggestions for improvement will be gratefully received.

CAMBRIDGE,
March 28, 1889.

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ERRATA.

p. 80, l. 16, *for* 132 *read* 133.

p. 156, l. 10, *for* S *read* O.

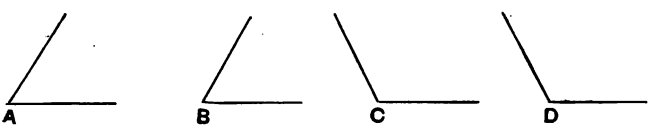
p. 264, l. 8, *for* $\theta = \frac{1}{2}n\pi + a$ or $\frac{1}{2}(ln+1)\pi - a$ *read* $\theta = n\pi + a$.

CHAPTER I.

INTRODUCTION.

§ 1. The Trigonometrical Angle.

1. In ordinary Geometry, an angle is entirely determined by the *relative position* of its two bounding straight lines. Thus, it is represented in a figure by two straight lines meeting at a point.



Two such angles are said to be equal if the pair of lines—supposed to be rigidly connected—which bound one angle may be superposed exactly on the pair of lines which bound the other angle. Thus, in the accompanying figure, the acute angles A and B are equal: and the obtuse angles C and D are equal.

2. But in Trigonometry we consider the *mode* in which an angle is *described*.

An angle is described by the revolution of a straight line about one of its extremities which is fixed.

Thus, the hand of a clock describes an angle by revolving about the centre of the clock-face.

The lines which *bound* a geometrical angle are those *between* which the revolver moves in describing the corresponding trigonometrical angle.

The trigonometrical angle is *measured* by the **Amount of Revolution** which the revolver undergoes in its movement between the two bounding lines.

3. A line which has revolved continuously from any position back to that same position is said to have described a *Complete Revolution.*

Thus, the minute-hand of a clock describes a complete revolution in an hour : the hour-hand in 12 hours.

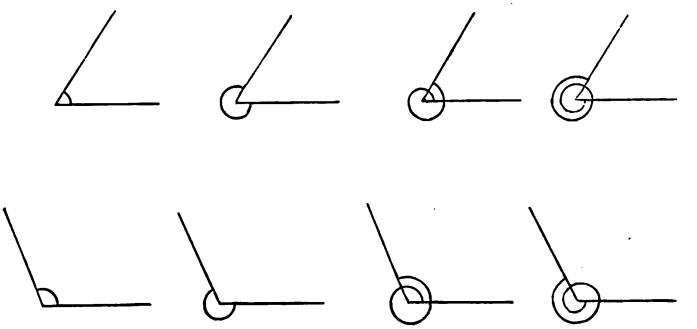
The *quarter* of a complete revolution is called a *quadrant.* This angle is the same as the geometrical right-angle. In describing a complete revolution, the Quadrants are called the first, second, third, and fourth, respectively, *in the order in which the revolver passes through them.*

4. To every *Geometrical angle* (see Fig. of Art. 1) there corresponds an indefinite number of *Trigonometrical angles.* For there is an indefinite number of ways by which the revolver may be supposed to have moved from the one to the other of the two lines which bound the Geometrical angle.

(1) The revolver may begin to revolve towards *either side* of one of the bounding lines to reach the other.

(2) The revolver may describe *any number of complete revolutions* before coming to rest.

Thus, to represent in a Figure the *Trigonometrical angle,* we must indicate the *side of the bounding lines* on which the



angle is formed, and the *number of complete revolutions* which have been described.

A part of a small spiral terminated by the two bounding lines may be used to indicate the *Amount of Revolution.*

Thus, the four acute angles in the accompanying figure are geometrically equal, but the spirals indicate in each case different *amounts of revolution.*

Similarly as regards the four obtuse angles.

The position of the minute-hand of a clock gives us the *geometrical angle* between its present position and its position at the beginning of the hour. But it is the position of the hour-hand which tells us how many complete revolutions the minute-hand has described since 12 o'clock. Thus the minute-hand and the hour-hand *combined* give the measure of the *trigonometrical angle* which the minute-hand has described. The *way* of revolution is of course indicated by the order of the numbers on the circumference of the clock-face.

5. Sometimes it is necessary to distinguish between the line from which the revolver *starts* and the line at which the revolver *stops* in describing any angle. The first is called the **Initial Line** and the second the **Final Line.** The words *initial* and *final* thus refer to the beginning and end, respectively, of any particular revolution considered. The letters I and F will be used to indicate the initial and final lines bounding any angle.

6. From the above it follows that, whereas every *Geometrical angle* is [properly] less than two right-angles, a *Trigonometrical angle* may have any magnitude whatever. In fact, any Trigonometrical angle greater than two right-angles may be regarded as made up by the summation of a number of Geometrical angles each less than two right-angles. [Such an angle is contemplated in Euc. VI. 33.] And

Two trigonometrical angles are said to be equal, if the geometrical angles into which the one may be divided are equal, each to each, to those into which the other may be divided.

Subdivisions of the Right-angle.

7. The right-angle is subdivided in two ways according to the following tables.

Sexagesimal method.

1 Right-angle = 90 degrees, written 90°
1° i.e. 1 Degree = 60 minutes, written 60′
1′ i.e. 1 Minute = 60 seconds, written 60″

Centesimal method.

1 Right-angle = 100 grades, written 100^g
1^g i.e. 1 Grade = 100 minutes, written 100‵
1‵ i.e. 1 Minute = 100 seconds, written 100‶.

---

8. The advantage of the centesimal method is that it is a mere application of the decimal system of notation. Thus

$$\begin{aligned} 2 \text{ rt. } \angle \text{ s } 35^g\ 27‵\ 25‶{\cdot}4 &= 2 \text{ rt. } \angle \text{ s } 35^g\ 27‵{\cdot}254 \\ &= 2 \text{ rt. } \angle \text{ s } 35^g{\cdot}27254 \\ &= 2{\cdot}3527254 \text{ of a rt. } \angle . \end{aligned}$$

Similarly $9^g\ 3‵\ 1‶{\cdot}73 = {\cdot}09030173$ of a rt. $\angle$.

In reducing angles so expressed to decimals of a right-angle, we have only to transform the integral numbers into decimals, taking care to express the *units* of each kind as *hundredths* of the next higher subdivision.

9. Observe that a complete revolution = 4 right angles = 360°, and that half a complete revolution = 2 right angles = 180°.

The student must be careful to avoid confusing the *three* different uses of the words minute and second.

The English or sexagesimal angular minute is a *sixtieth* of an angular *degree*.

The French or centesimal angular minute is a *hundredth* of an angular *grade*.

The minute of *Time* is a *sixtieth* of an *hour*.

Similarly as regards the *second*.

10. *To reduce an angle from the* Sexagesimal *to the* Centesimal *measurement.*

Reduce the angle to the decimal of a right-angle, and mark off the grades &c. as in Art. 8.

Ex. 17° 11′ 22″·98.

60) 22·98 seconds

60) 11·383 minutes

90) 17·189716̇ degrees

·190996851̇ right-angle

Answ. 19ᵍ 9ᶜ 96ᶜᶜ·851̇.

11. *To reduce an angle from the* Centesimal *to the* Sexagesimal *measurement.*

Mark off the angle as a decimal of a right-angle as in Art. 8, and reduce to degrees &c.

Ex. 19ᵍ 9ᶜ 96ᶜᶜ·851̇

·190996851̇(851 &c.) right-angle

90

17·189716̇(66659 &c.) degrees

60

11·3829̇(96 &c.) = 11·383 minutes

60

22·98 seconds.

Answ. 17° 11′ 22″·98.

12. In all tables of reference, records of observation &c. angles are expressed in terms of the sexagesimal subdivisions of the right-angle. The immense labour which would be involved in reducing these to the centesimal system has not yet been undertaken. Hence the centesimal subdivisions, though introduced by the French at the beginning of the century, have performed no service except that of supplying elementary exercises in Reduction for the Trigonometrical student.

13. Angles which together make up *one* right-angle are said to be **Complementary** to one another: and each angle is called the **Complement** of the other.

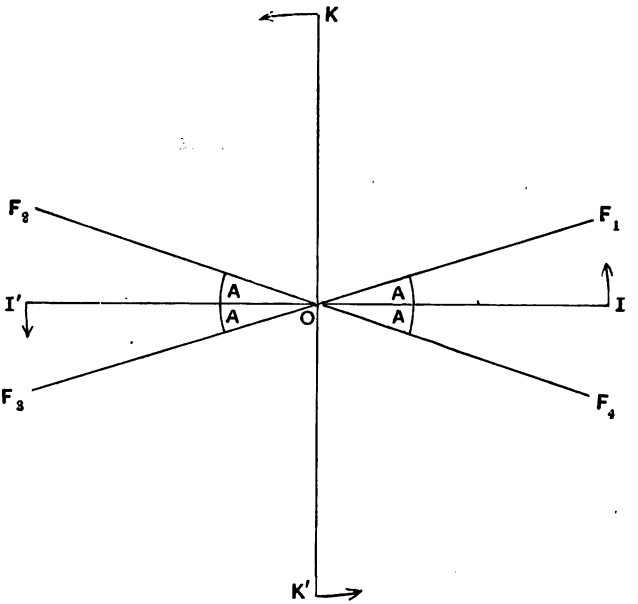
Thus, the pairs of angles 30° and 60°, 47° 13′ 29″ and 42° 46′ 31″ are complementary pairs. If A denote any angle, $90° - A$ denotes its complement.

Angles which together make up *two* right-angles are said to be **Supplementary** to one another: and each angle is called the **Supplement** of the other.

Thus, the pairs of angles 30° and 150°, 47° 13′ 29″ and 132° 46′ 31″ are supplementary pairs. If A denote any angle, $180° - A$ denotes its supplement.

The trigonometrical in terms of the corresponding geometrical angle.

14. Let ΓOI, $K'OK$ be two lines at right-angles cutting in O.



Let a line revolve from OI, *as initial position*, towards the perpendicular OK. Then, when the revolver

is between OI and OK, it is in the 1*st* Quadrant.
when ..... OK and OI', ... 2*nd* .....
when ..... OI' and OK', ... 3*rd* .....
when ..... OK' and OI, ... 4*th* .....

15. *To find the general expression for the trigonometrical angle, when the* final *position of the revolving line is given.*

Let OF be the final position of the revolving line corresponding to any angle.

1. *Let* OF_1 *be in the* 1*st Quadrant.*

Let the *acute* angle IOF_1 be A.

Then, the revolving line may have described any number of complete revolutions, bringing it back to OI, before finally resting at OF_1.

Hence the trigonometrical angle IOF_1 is $n \cdot 360° + A$ where n is any whole (positive) number.

2. *Let* OF_2 *be in the* 2*nd Quadrant.*

Let the acute angle $I'OF_2$ be A.

Then, any number of complete revolutions, i.e. $n \cdot 360°$, may be first described, bringing the revolver back to OI.

Now $IOF_2 + I'OF_2 = 180° \therefore IOF_2 = 180° - I'OF_2 = 180° - A$.

Thus the trigonometrical angle IOF_2 is

$$n \cdot 360° + 180° - A = (2n + 1)\ 180° - A.$$

3. *Let* OF_3 *be in the* 3*rd Quadrant.*

Let the acute angle $I'OF_3$ be A.

Then, the revolving line may be supposed

first, to make n complete revolutions from OI back to OI;

then, to revolve from OI to OI', through 180°;

and *finally*, to revolve from OI' to OF_3, through angle A.

Thus, the trigonometrical angle IOF_3 is

$$n \cdot 360° + 180° + A = (2n + 1)\ 180° + A.$$

4. *Let* OF_4 *be in the* 4*th Quadrant.*

Let the acute angle IOF_4 be A.

Then, the revolving line may be supposed

first, to make $(n-1)$ complete revolutions from OI back to OI;

then, to revolve from OI to OI', through 180°;

and *finally*, to revolve from OI' to OF_4, through $180° - A$.

Thus, the trigonometrical angle IOF_4 is

$$(n-1)\,360° + 180° + 180° - A = n\,.\,360° - A\,*.$$

16. By the last article, we see that we may express any *trigonometrical* angle in terms of the *geometrical acute* angle which the final position of the revolving line makes with the initial position (produced if necessary).

Summing up the results, when the final line is in

1st Quad., trig. angle = even multiple of 180° + an acute angle.
2nd Quad., trig. angle = odd multiple of 180° − an acute angle.
3rd Quad., trig. angle = odd multiple of 180° + an acute angle.
4th Quad., trig. angle = even multiple of 180° − an acute angle.

These results should be carefully noted and remembered†.

§ 2. Areas.

17. In order to measure an *area*, it is necessary to express the ratio it bears to some *known area*; just as to measure a length or a weight, it is necessary to express the ratio it bears to a known length or weight.

* In these four results, n is understood to be 0 or some *positive* integer. An explanation will be given later according to which n may be positive or *negative*.

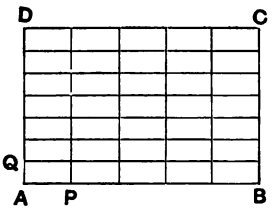
† The student should observe that to speak of an *angle* as being in a certain Quadrant is misleading: it is the *revolving line* which passes through the several Quadrants, and whose final position in one or other of the Quadrants indicates the magnitude of the angle.

18. *To find the area of a* rectangle, *when the lengths of its adjacent sides are known.*

Let $ABCD$ be a rectangle.

Let AB contain m parts, each equal to AP; and let AD contain n parts, each equal to AQ.

From the points of division draw lines parallel to the sides of the rectangle.



Then the whole rectangle $ABCD$ is divided into small rectangles—the adjacent sides of each of these rectangles being equal to AP and AQ.

And the number of these rectangles is $m \times n$.

Thus,

rect. whose adj. sides are m *times* AP and n *times* AQ, $= mn$ *times* rect. whose adj. sides are AP and AQ.

19. In the above proposition, suppose that $AP = AQ$. Then the rectangles, into which $ABCD$ is divided, are *squares.*

Let AP or AQ be taken as the *unit of length.* Then, the measure of AB in terms of this unit is m, and the measure of AD in terms of this unit is n.

If we take the square on the unit of length as the *unit of area*, then the measure of $ABCD$ in terms of this unit is mn.

Thus, the measure of the **area** of a rectangle is equal to the **product** of the measures of the lengths of its two adjacent sides.

20. The student should observe that an *area* is measured by the *product of* **two** *numbers* which measure *lengths.*

This is expressed by saying that *area* is of the *second dimension* in length.

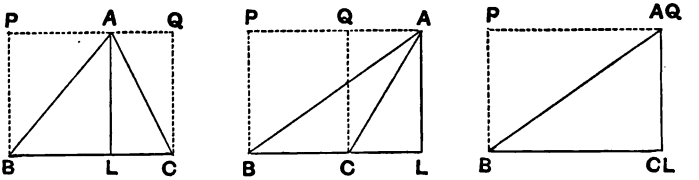
21. The conclusion of Art. 19 may be expressed more briefly thus:—

The area of a rectangle = product of the lengths of its sides.

But arithmetically we can only multiply a quantity by an abstract number. If we extend the meaning of the term multiplication beyond that understood in Arithmetic, we may speak of physically multiplying one physical quantity by another, whenever the arithmetical multiplication of the *measures* of these quantities gives us the measure of some other physical quantity depending on these. It should be clearly understood that we are in this case *extending* the meaning of the word multiplication.

The area of a triangle.

22. By Euc. I. 41, if a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram is double of the triangle.



Now, let ABC be any triangle,—acute-angled, obtuse-angled or right-angled.

From the vertex A, draw AL perpendicular to the base BC produced if necessary.

If the angle C is acute, L will fall between B and C,
if ..... obtuse, L will fall beyond C,
if ..... right, L will fall on C.

Through A draw PAQ parallel to BC; and draw BP, CQ parallel to LA.

Then, in each case, the triangle ABC and the parallelogram $PBCQ$ are on the same base BC, and between the same parallels BC, PQ,

$\therefore$ the $\triangle$ ABC is $\frac{1}{2}$ the parallelogram $PBCQ$.

But the rectangle $PBCQ$ is measured by the product of the measures of BC and BP; i.e. of BC and AL,

$\therefore$ the $\triangle$ $ABC = \frac{1}{2}$ the product BC, AL
$= \frac{1}{2}$ (base $\times$ altitude).

23. The student should compare the algebraical and the geometrical uses of the word *square*. In Geometry a square is a figure of a certain shape. In Algebra, a square is a product of two equal numbers. Now, by Art. 19, the measure of the area of a *square* = the product of the *two equal* measures of its adjacent sides. It might be supposed that this explains the identity of names in Algebra and Geometry. But this is only a partial explanation. In the above article, we had arbitrarily taken a *square* as our unit of area. If we had taken some other figure as unit of area, a figure of a corresponding shape would have been represented by the product of two equal numbers. If, e.g., the unit of area had been a *triangle whose base and altitude were equal*, then any other triangle whose base and altitude were equal would have been measured by the product of two equal numbers [and any triangle would be the product, (instead of $\frac{1}{2}$ the product,) of its base and altitude]. It is, thus, purely *arbitrary*, but (as the student may see on reflection) highly convenient, to choose a *square* as the unit of area. *Whatever unit of area had been chosen, any other area would have been proportional to the product of two lengths.* This is equivalent to Euclid's statement that "Similar rectilineal figures are to one another in the *duplicate* ratio of their homologous sides."

24. The area of any rectilineal figure may be found by dividing it into triangles: and applying the formula,

$$\text{triangle} = \tfrac{1}{2} \text{ (base} \times \text{altitude)}.$$

25. Since

$$\text{triangle} = \tfrac{1}{2} \text{ (base} \times \text{altitude)},$$

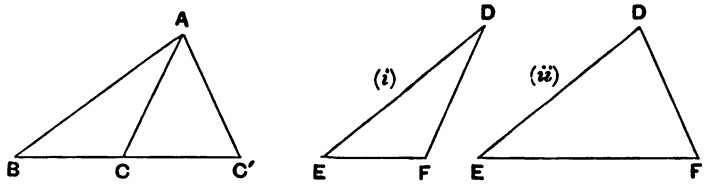
$\therefore$ triangles of equal altitudes are to one another as their bases.

This is Prop. 1 of Euc. Book VI. The student, who has not read Book VI. is advised, here, to read and learn Props. 2, 4, 5, 6 and 7 of that book. These props. are necessary for a great deal of what follows.

§ 3. Miscellaneous Propositions.

Equality in all respects of two triangles.

26. In Euc. I. (4, 8, 26) is proved the equality in all respects of two triangles, *three* elements of which in each case are *given* equal. Euclid omits one case, which may be now considered.



Let ABC, DEF be two triangles, having the two sides AB, AC equal to the two sides DE, DF, each to each; and likewise the angle ABC, opposite to AC, equal to the angle DEF, opposite to DF. The angles ACB, DFE opposite the other equal sides AB, DE shall be either equal or supplementary.

(i) *If EF and BC are equal*, then, by Euc. I. 4, the triangles ABC, DEF are equal in all respects. But (ii) *if not*, one of them must be the greater. Let EF be the greater. Produce BC to C', so that $BC' = EF$, join $C'A$. Then, by Euc. I. 4, the triangles ABC', DEF are equal in all respects.

But $\because AC = AC'$, $\therefore \angle ACC' = \angle AC'C$. [Euc. I. 5.]

And the angles ACB, ACC' are supplementary,

$\therefore$ the angles ACB, $AC'C$ are supplementary,

i.e. the angles ACB, DFE are supplementary,

$\therefore$ in case (i) ACB, DFE are equal,

and in case (ii) ACB, DFE are supplementary. Q.E.D.

Cor. 1. Since of two supplementary angles, not equal to one another, one must be acute and the other obtuse, therefore if, in the above proposition, we know

either that *both* the angles ACB, DFE are acute,
or that *both* the angles ACB, DFE are obtuse,
or that *one of* the angles ACB, DFE is right,

then it follows that ACB, DFE are necessarily equal without any other alternative.

Cor. 2. If AC is equal to or greater than AB, then $\angle ABC$ is equal to or greater than $\angle ACB$.

But two angles of a triangle are together less than two right-angles, therefore $\angle ACB$ is acute.

Similarly DF being equal to AC, and DE to AB, the angle DFE is acute.

This brings us to Cor. 1, showing that the angles ACB, DFE are necessarily equal.

27. The proposition of the last article is called *The Ambiguous Case* of the equality of two triangles. The ambiguity is removed by the corollaries. Further consideration of the matter is postponed to the chapter on Solution of Triangles.

28. We see from Art. 26, that there are *four* cases in which the exact equality of two triangles may be shown. But the last case is Ambiguous.

In each case, a *side* is given equal.

The three elements given equal are in

Case I. *Two angles and a side.* Euc. I. 26.

Case II. *Three sides.* Euc. I. 8.

Case III. *Two sides and the angle included by them.* Euc. I. 4.

Case IV. *Two sides and the angle opposite one of them.* Art. 27.

These four cases correspond, respectively, to the four propositions 4, 5, 6, 7 of Book VI. in which the *similarity* of two triangles is shown.

Case I. *Two angles* given equal. Euc. VI. 4.

Case II. *Three sides* given proportional. Euc. VI. 5.

Case III. *Two sides* given proportional, and *the angle included by them* given equal. Euc. VI. 6.

Case IV. *Two sides* given proportional, and *the angle opposite one of them* given equal. Euc. VI. 7.

29. By Euclid I. 32, the angles of a triangle are together equal to two right-angles.

But 1 right-angle = 90 degrees or 90°,

∴ 2 right-angles = 180 degrees or 180°,

∴ the angles of a triangle are together equal to 180°.

Example. The angles of a triangle are as 2 : 3 : 4. *Find them.*

We have to divide 180° into 3 parts, which are in the proportion 2 : 3 : 4.

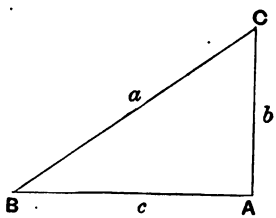
The required parts are

$$(1)\quad \frac{2}{2+3+4} \text{ of } 180° = \frac{2}{9} \text{ of } 180° = 40°,$$

$$(2)\quad \frac{3}{2+3+4} \text{ of } 180° = \frac{3}{9} \text{ of } 180° = 60°,$$

$$(3)\quad \frac{4}{2+3+4} \text{ of } 180° = \frac{4}{9} \text{ of } 180° = 80°.$$

30. By Euclid I. 47, the square on the hypothenuse of a right-angle is equal to the sum of the squares on the two sides containing the right-angle.



Let ABC be a right-angled triangle, right-angle at A.

If a, b, c are the measures of the sides opposite to A, B, C, respectively; then a^2, b^2, c^2 are the measures of the squares on these sides, respectively,

$$\therefore\ a^2 = b^2 + c^2.$$

By this equation, we can find *any one side of a right-angled triangle when the other two are given.*

Example 1. *The sides of a right-angled triangle are* 5 *ft. and* 12 *ft. respectively, find the hypothenuse.*

Take a foot as the unit of length.

Then 5 and 12 are the measures of the two sides.

∴ (measure of hypothenuse)$^2 = 5^2 + 12^2 = 25 + 144 = 169$,

∴ measure of hypothenuse $= \sqrt{169} = 13$,

∴ hypothenuse = 13 ft.

Example 2. *The hypothenuse and base of a right-angled triangle are* 2 *ft.* 5 *in. and* 1 *ft.* 9 *in. respectively. Find the altitude.*

Take an inch as unit of length. Let x measure altitude. Then, we have,

$$29^2 = 21^2 + x^2.$$

$$\therefore\ x^2 = 29^2 - 21^2 = 841 - 441 = 400,$$

$$\therefore\ x = 20,$$

$$\therefore\ \text{altitude} = 20 \text{ in.} = 1 \text{ ft. } 8 \text{ in.}$$

EXAMPLES I.

1. Express in grades, minutes and seconds the following decimals of a right-angle.

(1) ·000003, (2) ·0025, (3) ·04, (4) ·356, (5) 1·888, (6) 234·25789, (7) 52·04065042003.

2. Express the above angles in degrees, minutes, seconds.

3. Express as decimals of a right-angle:—

(1) 4ᶜ, (2) 70ᶜ, (3) ·04ᵍ, (4) 5ᵍ 2ᶜ 45ᶜᶜ, (5) 26ᵍ 46ᶜ 35ᶜᶜ·62, (6) 32ᶜᶜ·4, (7) 3ᶜᶜ·24, (8) 0ᶜᶜ·324, (9) 11° 4′ 12″, (10) 31° 3′, (11) 61° 1′ 12″.

4. Express in the centesimal method:—

(1) 1° 8′ 8″·8, (2) 3° 4′ 7″·2, (3) 61° 1′ 12″.

5. Express in the sexagesimal method:—

(1) 3ᵍ 33ᶜ 33ᶜᶜ·3, (2) 45ᵍ 5ᶜ 3ᶜᶜ, (3) 7ᵍ 8ᶜ 79ᶜᶜ.

6. Show that the number of degrees contained in any angle is $\frac{9}{10}$ of the number of grades contained in the same angle.

7. If an angle is expressed as a decimal of a degree, show that it may be expressed as a decimal of a grade by removing the decimal point one place to the right and dividing by 9.

8. If an angle is expressed as a decimal of a grade, show that it may be expressed as a decimal of a degree by subtracting from the given decimal the decimal formed from it by removing the point one place to the left.

9. The angles of a triangle are in arithmetical progression, and the greatest of them is $\frac{5}{3}$ of the least. Find them.

10. The number of degrees in an angle is to the number of grades in its complement as 3 is to 5. Find the angle.

11. A, B, C are the three angles of a triangle. The number of degrees in A is equal to the number of grades in B and to the product of the number of degrees and of grades in C. Find A, B, C.

12. The side AB of a triangle ABC is produced to D, and AE is drawn making an angle with AC equal to BCA which is less than BAC. The angle CBD is 112° 11′ 44″; and the angle EAB is 67° 48′ 16″. Find the angles of the triangle ABC.

13. What trigonometrical angles has the minute hand of a clock described since midnight at 3.15 a.m., at 7.40 p.m., and at 9.10 a.m., respectively?

14. $ABCD$ is a square, AC and BD are its diagonals. Find the general expressions for the trigonometrical angles bounded by the lines

(1) AB and AC, (2) AB produced and BD.

15. Find in degrees the angles of

(1) an equilateral triangle, (2) a regular hexagon, (3) a regular pentagon, (4) a regular dodecagon, (5) a regular quindecagon.

16. Find the hypothenuse of the right-angled triangles whose sides are respectively,

(1) 3 ft. and 4 ft. (2) 2 ft. 9 in. and 4 ft. 8 in. (3) 4 ft. and 4 ft. 7 in.

17. Find the third side of the right-angled triangles whose hypothenuse and other side are, respectively,

(1) 26 in. and 10 in. (2) 4 ft. 5 in. and 2 ft. 4 in. (3) 7 ft. 1 in. and 3 ft.

18. The sides of a right-angled triangle are $a^2 - b^2$ and $2ab$ respectively. Find the hypothenuse.

19. OA is a line of any length. AB is perpendicular to OA, BC to OB, CD to OC, DE to OD &c., and

$$AB = BC = CD = DE = \&c. = OA.$$

Find the lengths of the lines OB, OC, OD, OE &c. in terms of OA.

20. Show in the method of Euclid, Book I., that

(1) Two triangles, which are equal in area and have a side and adjacent angle of one equal to a side and adjacent angle of the other, are equal in all respects.

(2) Two triangles, which are equal in area and have two angles of the one equal to two angles of the other, each to each, are equal in all respects.

(3) Two triangles, which are equal in area and have a side and opposite angle of one equal to a side and opposite angle of the other, are equal in all respects.

(4) If two triangles, which are equal in area, have two sides of the one equal to two sides of the other, each to each, then the angle contained by the two sides of the one is equal or supplementary to the angle contained by the two sides of the other.

CHAPTER II.

THE ANGLE AND THE CIRCLE.

§ 1. On Measurement.

31. Two quantities are said to be **homogeneous** or **comparable**, when one is either equal to, less than, or greater than the other.

Thus, an inch and a mile—a degree and a grade—a triangle and a square—are pairs of quantities which are *comparable* in magnitude. But a line and an angle—a line and an area—an angle and an area—are pairs of quantities *incomparable* in magnitude.

Two homogeneous quantities are said to be **commensurate**, when the multiplication of the smaller by some *finite* number gives a quantity greater than the greater.

Thus, an inch and a mile are *commensurate;* but the space occupied by an atom and that occupied by the Solar System are practically *incommensurate.*

Two homogeneous commensurate quantities are said to be **commensurable**, when both are exact multiples of some common quantity commensurate with each.

Thus, $3\frac{1}{2}$ inches and $17\frac{1}{3}$ inches are *commensurable;* for $3\frac{1}{2}$ inches $= 21$ *sixths-of-an-inch* and $17\frac{1}{3}$ inches $= 104$ *sixths-of-an-inch.* Also $3\sqrt{2}$ inches and $5\sqrt{2}$ inches are *commensurable.* But 1 inch and $\sqrt{2}$ inches are *incommensurable:* for, since there is no fraction with integral numerator and denominator exactly equal to $\sqrt{2}$, 1 inch cannot be divided into any finite number of parts such that a number of those parts shall exactly equal $\sqrt{2}$ inches*.

* The distinction between *commensurable* and *incommensurable* is purely theoretical. Practically we cannot compare the measures of two quantities exactly; hence we can never determine whether they are really commensurable or not.

32. The definitions of the last article are equivalent to the following:

Two quantities are *comparable*, when the one has *some* ratio to the other: they are *commensurate*, when the ratio of either to the other is *finite*: they are *commensurable*, when their ratio is measured by a fraction *with finite integral numerator and denominator.*

33. A quantity of any kind is *measured* by assigning the *ratio* it bears to some known quantity of the same kind called the *unit.*

Thus we measure *lengths* by assigning their ratio to (say) *a foot;* we measure *angles* by assigning their ratio to (say) *a right angle.*

The ratio of the given quantity to the unit is called the *abstract or numerical measure* of the quantity; or sometimes, more shortly, *the measure of the quantity.*

34. An *accurate* measurement of any quantity is practically impossible.

Suppose that we know that a magnitude lies between $m-1$ units and $m+1$ units. Then the error involved in taking m units for the true value is *less than* 1 *unit.* We should, therefore, say that *to within* 1 *unit* its measure is m units.

If another magnitude has to be found by multiplying the above by any number k, this second magnitude lies between $km-k$ units and $km+k$ units. The error involved in taking km units for the true value is *less than* k *units.* We should, therefore, say that *to within* k *units* its measure is km units.

Therefore, the possible error in assigning the measure of a magnitude by multiplying the measure of another magnitude by k is increased k-fold. The ratio of the possible error to the whole measure is as before, viz. $\frac{1}{m}$.

If now the number k is fractional and we wish to find an *integral* number of units in our product, we need not calculate

k more closely than to *within* $\frac{1}{m}$ *of unity*. For, by the same reasoning as above, provided the error in k is less than $\frac{1}{m}$, the error in mk will be less than 1.

Example 1. The side of a square is, *to within an inch*, 57 yd. 2 ft. 3 in. Find the diagonal and assign the accuracy of the result.

57 yd. 2 ft. 3 in. = 2079 in.

$\therefore$, by Euc. I. 47, (diagonal)^2 = (2079 in.)^2 + (2079 in.)^2

= 2 × (2079)^2 sq. in.

$\therefore$ diagonal = $\sqrt{2}$ × 2079 in.

Now $\sqrt{2} = 1{\cdot}4142$ &c. This differs from $1{\cdot}414$ by less than $\cdot 0003$, i.e. less than $\frac{1}{3000}$.

$\therefore$ in finding an integral number of inches in the diagonal, we may use $1{\cdot}414$ for $\sqrt{2}$.

$\therefore$ diagonal = $1{\cdot}414 \times 2079$ in. = $2939{\cdot}706$ in.

= 2940 in. *or* 81 yd. 2 ft. *nearly*.

Since the original possible error was 1 inch, the possible error in the result is $\sqrt{2}$ inches, i.e. about $1\frac{1}{2}$ in.

Example 2. The ratio of the circumference to the diameter of a circle being $3{\cdot}14159$ &c., find the circumference of a circle whose diameter is, *to within an inch*, 13 yd. 8 in.

13 yd. 8 in. = 476 in.

Now $3{\cdot}14159$ &c. differs from $3\frac{1}{7}$ by less than $\cdot 002$ or $\frac{1}{500}$,

$\therefore$ in finding an integral number of inches in the circumference we may use $3\frac{1}{7}$ instead of $3{\cdot}141$ &c.

$\therefore$ circumference = $\frac{22}{7}$ of 476 in. = 41 yd. 1 ft. 8 in.

The result is correct to within about 3 inches.

Fundamental Proposition on Incommensurables.

35. Let there be two quantities which vary together, in such a way that to any value of one there corresponds a value of the other, and that when either increases the other increases. Then, if the ratio of any two *commensurable* values of the one is equal to the ratio of the two corresponding values of the other, the same will hold for *all* pairs of values whether commensurable or incommensurable.

Let x and y represent the two varying quantities. Let subscript numbers indicate particular values, the *same* subscript number being used for *corresponding* values.

We have to show that

$$x_1 : x_2 = y_1 : y_2$$

where x_1 and x_2 are incommensurable values of x.

If possible, let $x_1 : x_2 = y_1 : y_3$ (1)

where $y_3 > y_2$ and $\therefore x_3 > x_2$.

Suppose x_1 to be divided into a number m of parts, each equal to a, where $a < x_3 - x_2$.

Then, if na is the largest multiple of a less than x_3,

$$na < x_3 \text{ but } na > x_2.$$

Call na, x_4: then, since $x_4 < x_3$, $\therefore y_4 < y_3$.

Again, since x_1 and x_4 (viz. ma and na) are commensurable,

$$\therefore x_1 : x_4 = y_1 : y_4 \quad (2).$$

Comparing (1) and (2),

since $x_4 > x_2 \ \therefore y_4 > y_3$.

But y_4 has been shown to be $< y_3$. Which is impossible. $\therefore x_1 : x_2$ is not unequal to $y_1 : y_2$ i.e. it is equal to it. Q.E.D.

§ 2. The Angle and the Circle.

36. A *circle* may be described by the revolution in a plane of a straight line about one of its extremities which remains fixed.

The fixed extremity is called the *centre*; the moving extremity traces out the *circumference*.

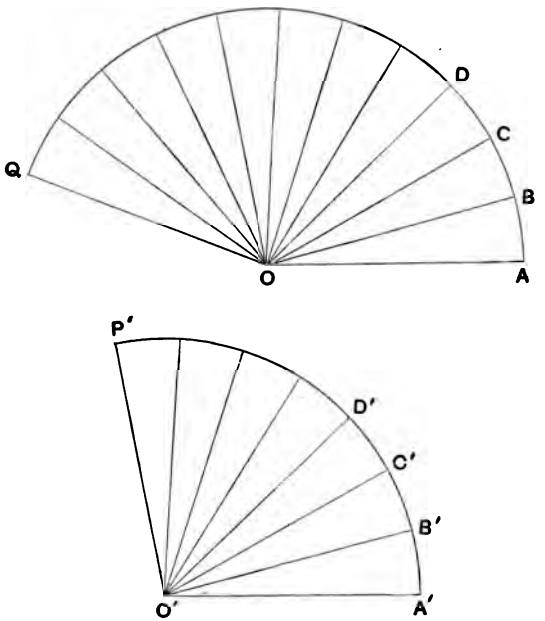
Any part of the *circumference* is called an *arc*; and any part of the area, traversed by the revolving radius, is called a *sector*.

Now the length of the arc and the area of the sector clearly depend on two elements, viz. (1) The *length* of the revolving radius. (2) The *amount of revolution* described.

Hence we see how naturally a *circle* is connected with *an angle* as described in the first chapter.

The two fundamental propositions connecting *angle*, *arc*, and *radius* will now be proved. The first proposition is proved in a manner intended to show the connexion between Euclid's method of treating proportionals and the algebraical method.

37. *In circles of* equal radii, *arcs which subtend* unequal angles *are to one another as the* angles.



Let the arcs AB, $A'B'$ of circles whose *radii* OA, $O'A'$ are *equal*, subtend angles AOB, $A'O'B'$ at the centres O, O' respectively. Then

arc AB : arc $A'B'$ *shall equal* angle AOB : angle $A'O'B'$.

If not, let the fractions which measure these ratios be unequal: and suppose that p/q is a fraction lying between them, so that (say)

$$\frac{\text{arc } AB}{\text{arc } A'B'} < \frac{p}{q} \text{ and } \frac{\text{angle } AOB}{\text{angle } A'O'B'} > \frac{p}{q}.$$

Then q . arc $AB <$ p . arc $A'B'$

and q . angle $AOB >$ p . angle $A'O'B'$.

At O describe q angles (including AOB) each equal to AOB, and at O' describe p angles (including $A'O'B'$) each equal to $A'O'B'$.

Then, the whole angle $AOQ = q$. angle AOB;

and the whole angle $A'O'P' = p$. angle $A'O'B'$,

$\therefore$ whole angle $AOQ >$ whole angle $A'O'P'$.

Again, by Euc. III. 26, the arcs AB, BC &c. which subtend equal angles are equal to one another.

$\therefore$ the whole arc $AQ = q$. arc AB,

and the whole arc $A'P' = p$. arc $A'B'$,

$\therefore$ whole arc $AQ <$ whole arc $A'P'$.

But again, by Euc. III. 26, since the radii OA, $O'A'$ are equal, and since angle $AOQ >$ angle $A'O'P'$,

$\therefore$ arc $AQ >$ arc $A'P'$. Which is absurd.

$\therefore$ arc AB : arc $A'B' =$ angle AOB : angle $A'O'B'$. Q.E.D.

Or the proposition may be proved as follows:

Using the same figure, but considering the angles AOB, $A'O'B'$ to be *equal*, it follows at once from Euc. III. 26, that

arc AQ : arc $A'P' =$ angle AOQ : angle $A'O'P'$

i.e. the ratio of any two *commensurable* values of the arc is equal to the ratio of the corresponding values of the angle,

$\therefore$, by Art. 35, the same holds for *all* values.

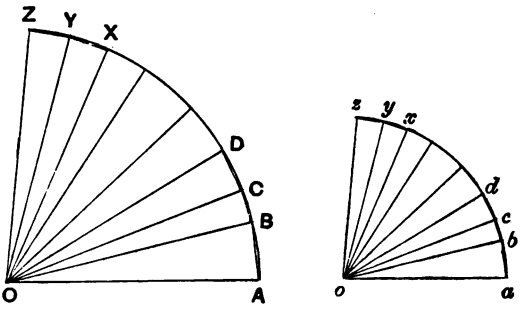
38. *In circles of* unequal radii, *arcs which subtend* equal angles *are to one another as the* radii.

Let the arcs AZ, az subtend *equal angles* at the centres O, o respectively. Then

arc AZ : arc az *shall equal* radius OA : radius oa.

Within angle AOZ, draw any number of radii OB, $OC \ldots OY$:

and at o, make the angles aob, $boc \ldots xoy$ equal to AOB, BOC, ... XOY, respectively. Then



$\because$ whole angle AOZ = whole angle aoz,

$\therefore$ remaining angle YOZ = remaining angle yoz.

Join AB, $BC \ldots YZ$; ab, $bc \ldots yz$.

Then $\because$ $AO : BO = ao : bo$, and $\angle AOB = \angle aob$,

$\therefore$ $\triangle AOB$ is similar to $\triangle aob$. [Euc. VI. 6.

$\therefore$ $AB : ab = AO : ao$.

Similarly $BC : bc = BO : bo = AO : ao$,

and $CD : cd = CO : co = AO : ao$.

.....

$\therefore$ $AB + BC + CD + \ldots\ldots : ab + bc + cd + \ldots\ldots = AO : ao$.

Now, as we increase the number of divisions of the arcs AZ, az, the rectilinear boundaries evidently approximate more and more closely to the arcs.

Hence ultimately we may assume that

$$AB + BC + CD + \ldots\ldots = \text{arc } AZ;$$

and

$$ab + bc + cd + \ldots\ldots = \text{arc } az.$$

$\therefore$ arc AZ : arc az = radius OA : radius oa*. Q.E.D.

39. The above proposition is of the highest importance.

* A stricter proof of this proposition will be given later.

Since *arc* and *radius* are both *lengths*, we may interchange the second and third terms of the proportion, and write

arc AZ : radius OA = arc az : radius oa,

when the angle AOZ = angle aoz.

In other words, the ratio *arc* to *radius* is constant for the same angle described at the centre of a circle.

40. PROP. I. *The ratio of the circumference to the diameter of a circle is the same for all circles.*

For, in the last article, suppose the angles AOZ, aoz to be each four right-angles. Now the arc subtended by four right-angles, i.e. by a complete revolution, is the whole circumference. Hence,

circumference AZ : radius OA = circumference az : radius oa.

$\therefore$ circumference AZ : diameter $2 . OA$

= circumference az : diameter $2 . oa$.

i.e. the ratio of the circumference to the diameter in *any one* circle is equal to the ratio of the circumference to the diameter in *any other* circle. Q.E.D.

PROP. II. *The angle which, at the centre of a circle*, subtends an arc equal to the radius, *is the same in magnitude for all circles.*

For, in the last article, suppose *the arc AZ to have been measured equal to the radius OA*, and the angle aoz to be still equal to the angle AOZ. Then, since

arc AZ : radius OA = arc az : radius oa.

$\therefore$ arc *az is also equal to* radius oa.

Thus, the angle, which, at the centre of *any one* circle, is subtended by an arc equal to the radius, is equal to the angle, which at the centre of *any other* circle, is subtended by an arc equal to the radius. Q.E.D.

41. From the above two propositions, follow two definitions.

1. The ratio of the circumference to the diameter of *any* circle is called π*.

* The Greek for *circumference* is περιφέρεια, of which π is the initial letter.

2. The angle, at the centre of *any* circle, subtended by an arc equal to the radius of the circle, is called a **Radian.**

The student must particularly observe that, since π is the ratio of one length to another length, π is an *abstract number:* and since the ratio of the circumference to the diameter is constant, π is of constant magnitude.

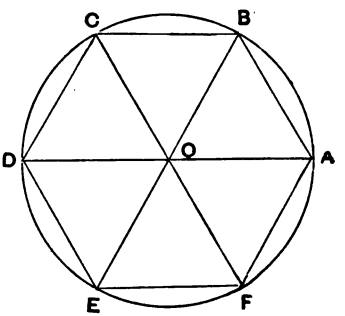
Also, since the angle, which at the centre of a circle subtends an arc equal to the radius, is constant in magnitude, therefore the *radian* is an *angle of fixed magnitude.*

42. Since $\frac{\text{circumference}}{\text{diameter}} = \pi$, in any circle.

$\therefore$ circumference $= \pi$. diameter $= 2\pi$. radius.

43. *To show that π is somewhat greater than* 3.

Let six angles AOB, BOC &c. each equal to $\frac{1}{6}$ of 4 right-angles, i.e. $\frac{1}{3}$ of 2 right-angles, be placed round the centre of the circle. Join AB, BC &c.



Then the three angles AOB, OBA, OAB being together equal to 2 right-angles, and the angle AOB being $\frac{1}{3}$ of 2 right-angles, the angles OBA, OAB are together equal to $\frac{2}{3}$ of 2 right-angles. And $OA = OB$,

$\therefore$ each of the angles OBA, OAB is $\frac{1}{3}$ of 2 right-angles,

$\therefore$ the triangle OAB is equilateral,

$\therefore$ each side of the hexagon $ABCDEF$ equals the radius,

$\therefore$ perimeter of the hexagon = 6 times the radius,
= 3 times the diameter,

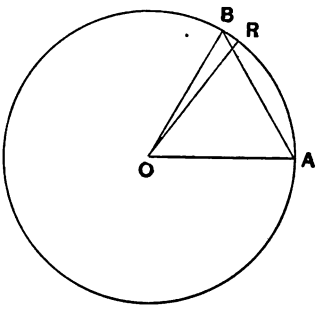
$\therefore$ circumference of the circle is somewhat larger than 3 times the diameter: or *ratio of circumference to diameter is somewhat greater than* 3.

44. *To show that the radian is somewhat smaller than the angle of an equilateral triangle.*

Suppose the arc AR to be measured equal to the radius OA, so that the angle AOR is the radian.

Then, the figure AOR is a *curvilinear* equilateral triangle.

If the curved side AR is bent into the straight line AB, we have a *rectilinear* equilateral triangle, showing that the radian angle AOR is somewhat smaller than the angle AOB of an equilateral triangle. Thus *the radian is somewhat less than* 60°.



45. *To find the relation between the radian and the right-angle.*

Since *angles* at the centre of a circle are to one another as the *arcs* which they subtend,

$$\therefore \frac{\text{two right-angles}}{\text{one radian}} = \frac{\text{arc subtended by two right-angles}}{\text{arc subtended by radian}}$$

$$= \frac{\text{semi-circumference}}{\text{arc equal to radius}} = \frac{\text{circumference}}{\text{diameter}} = \pi,$$

$$\therefore 2 \text{ right-angles} = \pi \text{ radians}.$$

46. The result of the preceding art. must not be forgotten.

Since π radians = 2 right-angles,

$$\therefore 1 \text{ radian} = \frac{2 \text{ right-angles}}{\pi}.$$

Now π is somewhat greater than 3, hence the radian is somewhat smaller than $\frac{2 \text{ right-angles}}{3}$, i.e. somewhat smaller than the angle of an equilateral triangle, as we have shown.

47. We have shown that π is somewhat greater than 3. Its value cannot be *exactly* expressed by any fraction having a finite numerator and denominator. In other words the circumference and diameter of a circle are *incommensurable*.

The value of π as far as the first 8 decimal places is 3·14159265...

Thus, if we use this value for π in determining the circumference of a circle from its diameter, our error will be less than one hundred millionth of the diameter. The student is advised to remember that $\pi = 3{\cdot}14159$ approximately.

$$\text{Hence 1 radian} = \frac{2 \text{ right-angles}}{\pi} = \frac{180^\circ}{\pi}$$

$$= \frac{180^\circ}{3{\cdot}14159265} \text{ approximately} = 57^\circ{\cdot}29577951.$$

Example 1. Show that as far as two places of decimals

$$\pi = \frac{22}{7} = 3\frac{1}{7}.$$

$$7\overline{)22{\cdot}000}\ \overline{3{\cdot}142}\ \&c. \qquad \text{Hence } \pi = 3\frac{1}{7}.$$

Example 2. Show that, correctly to one millionth, $\pi = \frac{355}{113}$.

Dividing 355 by 113, the quotient is 3·1415929 &c.

Example 3. Find the number of radians in 15^0.

We have $180^0 = \pi$ radians,

$$\therefore 15^0 = \frac{1}{12} \text{ of } 180^0 = \frac{\pi}{12} \text{ radians} = {\cdot}261799 \text{ radians}.$$

Example 4. Find the number of degrees in 11 radians.

We have π radians $= 180^0$,

$$\therefore \text{ 1 radian } = \frac{180^0}{\pi} = 57^0{\cdot}29578,$$

$$\therefore \text{ 11 radians} = 630^0{\cdot}25358.$$

Circular Measure of Angles.

48. Since the *radian* is an angle of invariable magnitude, it

may be taken as the *unit* angle in terms of which other angles may be measured.

Def. The ratio of any angle to the radian is called the **circular measure** of the angle. In other words, *the radian is the unit of circular measurement.* Thus, since $180° = \pi$ radians,

$\therefore$ circular measure of $180°$ is π.

Similarly the circular measure

of $90°$ is $\frac{\pi}{2}$; of $60°$, $\frac{\pi}{3}$; of $45°$, $\frac{\pi}{4}$; of $30°$, $\frac{\pi}{6}$, &c.

49. Using the symbol ^{c} to denote the radian, or the unit of circular measurement, we have

$$2 \text{ right-angles} = 180° = 200^g = \pi^c.$$

Thus the *same* angle is expressed severally by the numbers 2, 180, 200, 3·14159, when we take the different units—right-angle, degree, grade, radian—respectively.

By the above system of equations, we may always find the number of degrees or of grades in an angle whose circular measure is given; or conversely.

Example. Find the number of degrees in the angle whose circular measure is θ.

We have
$$\pi^c = 180°,$$
$$\therefore 1^c = \frac{180°}{\pi},$$
$$\therefore \theta^c = \frac{180 . \theta°}{\pi},$$
$$\therefore \text{the number of degrees required is } \frac{180 . \theta}{\pi}.$$

50. The circular measure of angles, i.e. the measure when the radian is taken as unit, is so frequently used in theoretical investigations, that *the symbol for the unit is usually dropped.* But a *Greek* letter is generally used to denote such a measure of an angle: hence, the confusion arising is minimised.

Thus *the angle* θ is understood as an abbreviation for *the angle* θ *radians.* In particular, *the angle* π is short for π radians i.e. 180 degrees.

To avoid confusion the student should at first introduce the symbol ^{c} in any examples in which it is omitted.

51. The circular measure of an angle is the same as the ratio of the *arc*, subtended by the angle at the centre of a circle, to the *radius* of the circle.

Shortly, *circular measure of angle* $= \dfrac{\text{arc}}{\text{radius}}$.

For, the circular measure of any given angle means

$$\frac{\text{given angle}}{\text{radian angle}}.$$

But angles are to one another as the arcs they subtend.

$$\therefore \text{ circ. meas. of given angle} = \frac{\text{arc opp. given angle}}{\text{arc opp. radian}}$$

$$= \frac{\text{arc opp. given angle}}{\text{radius}}. \qquad \text{Q.E.D.}$$

52. If θ = circular measure of any angle, by above,

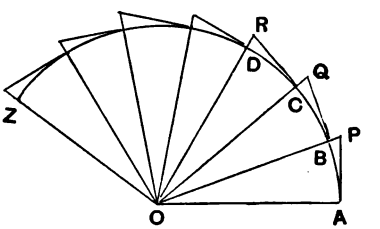
$$\frac{\text{arc}}{\text{radius}} = \theta,$$

$$\therefore \text{ arc} = \theta \times \text{radius}.$$

This result is important.

53. *The area of the* sector *of a circle is equal to that of a* triangle *having the* radius *and* arc *of the sector for* base *and* altitude *respectively.*

Let AZ be the arc and OA the radius of the sector AOZ.



Within the angle AOZ draw any number of radii OB, OC, &c. From the extremity of each radius, draw a perpendicular to it meeting the next radius produced. Then, by Euc. III. 16, these perpen-

diculars are the straight lines which are the closest possible to the arcs.

The $\triangle$s AOP, BOQ, &c. have each the same base equal to the radius; and the several altitudes AP, BQ, &c.

$\therefore$ the sum of these $\triangle$s = a $\triangle$ having its base equal to the radius and its altitude equal to the sum of the altitudes of the $\triangle$s.

But when the number of the $\triangle$s is increased, the sum of the heights AP, BQ, &c. approximates more and more closely to the arc AZ; and the sum of the $\triangle$s to the sector AOZ.

$\therefore$ ultimately,

area of sector = area of triangle having radius and arc for base and height respectively*.

54. If θ be the circular measure of any angle described at the centre of a circle of radius r, we have shown (Art. 52) that

$$arc \text{ opposite } \theta = \theta \,.\, r \qquad (1).$$

Again, any triangle $= \frac{1}{2}$ base $\times$ altitude,

$\therefore$ (by Art. 53) sector $= \frac{1}{2}$ arc $\times$ radius,

$\therefore$ *sector* containing $\theta = \frac{1}{2}\theta r^2$ (2).

The student should note and compare (1) and (2). In these equations, θ is an abstract number. Now, since *arc* is a *length*, it is proportional to *radius*, r. But since *sector* is an *area*, it is proportional to *square on radius*, r^2.

55. *To find the area of a circle.*

In the result of (2) last article, suppose that θ is the circular measure of four right-angles i.e. of a complete revolution. Then $\theta = 2\pi$.

But the sector containing *four right-angles* is the *whole* circle.

$\therefore$ area of circle $= \pi r^2$.

* A stricter proof of this proposition will be given later.

This equation should be noted and compared with the equation

length of circumference $= 2\pi r$.

56. From the equation *sector* $= \frac{1}{2}\theta r^2$ it follows that

(1) Sectors of circles of equal radii are to one another as the angles they contain. (Compare Art. 37.)

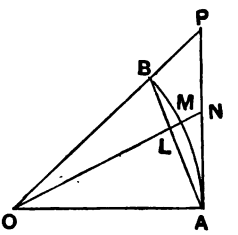
(2) Sectors of circles of unequal radii containing equal angles are to one another as the squares on their radii. (Compare Art. 38.)

(3) The circular measure of an angle is the same as the ratio of the sector containing it to the triangle whose base and altitude are each equal to the radius. (Compare Art. 51.)

More Rigid Proofs.

57. The *chord* of an arc is the straight line joining its extremities. The *tangent* of an arc, subtending an acute angle at the centre, will be here defined as "the perpendicular drawn from the extremity of one bounding radius and terminated by the other produced." In what follows, the angles considered will be *acute*. The results may easily be extended to angles of any magnitude.

58. PROP. I. The chord of an arc is entirely within and the tangent entirely without the circle. (Euc. III. 2, 16.)



Let AOB be any sector of a circle. From centre O, within the angle AOB, draw any straight line cutting the chord AB in L, the arc AB in M, the tangent AP in N.

$\because$ exterior $\angle OLB > \angle OAL$, i.e. $\angle OBA$ $\quad\therefore OB$, i.e. $OM, > OL$,

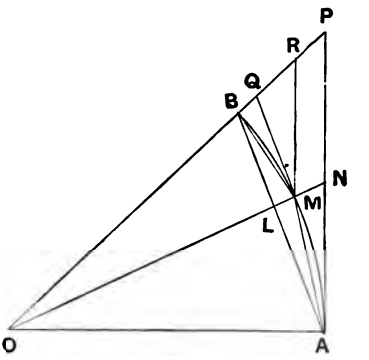
$\because$ right $\angle OAN > \angle ONA$ $\quad\therefore ON > OA$, i.e. OM,

$\therefore OM$ lies in magnitude between OL and ON,

$\therefore$ arc AMB lies in position between chord AB and tangent AP.

COR. Hence *the area of the sector AOB lies in magnitude between that of the triangles AOB and AOP.*

59. PROP. II. The arc of a circle lies in magnitude between its chord and its tangent.



For, in above figure, joining BM and AM,

sum of chords, AM and BM > chord AB.

And drawing from M, the tangent MQ, and MR parallel to NP,

then, the angle MQR being obtuse, $MQ < MR$.

But $MR : NP = OM : ON$,

$\therefore\ MR < NP$,

$\therefore$ *a fortiori* $MQ < NP$, and, adding AN,

sum of tangents, AN and MQ < tangent AP.

Now, the bent line AMB has more points in common with the arc AB than the straight line AB.

And, the two lines AN, MQ have more points in common with the arc AB than the single line AP.

Therefore, we may assume that

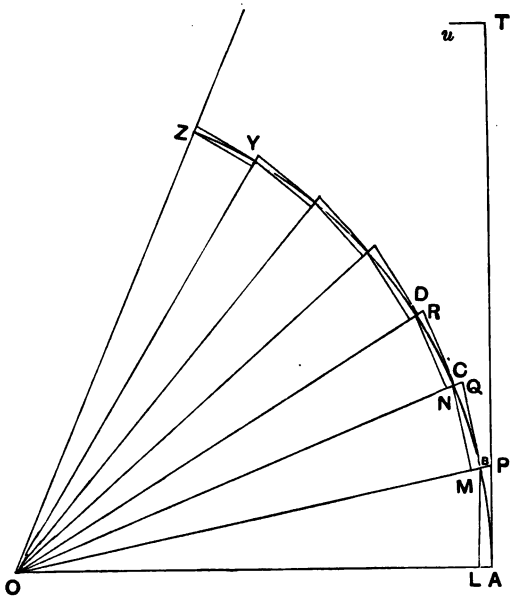
sum of chords AM and BM is nearer in magnitude than chord AB to the arc AB, and

sum of tangents AN and MQ is nearer in magnitude than tangent AP to the arc AB.

Combining these results with those above, it follows that *arc AB > chord AB*, and *arc AB < tangent AP*.

COR. *A fortiori*, arc $AB >$ perpendicular from B on OA.

60. Let AZ be an arc of a circle, centre O.



Suppose this arc divided at B, C, &c. and radii drawn to B, C, &c.

From each point of division,
drop perpendiculars, BL, CM, &c. on the preceding radii, and draw tangents AP, BQ, CR, &c. to meet the succeeding radii.

We shall refer to

BL, CM, &c. as the *Internal* Perpendiculars,
and AP, BQ, &c. as the *External* Perpendiculars.

And to

triangles OAB, OBC, &c. as the *Internal* Triangles,
and triangles OAP, OBQ, &c. as the *External* Triangles.

Also let AP, OZ produced meet in T.

61. PROP. III. The difference between the sums of the

external and internal triangles may be made less than any assigned area.

Let a triangle equal to the assigned area be described on AT as base. Let Tu be its height.

Suppose the arc to have been divided so that

AL, BM, &c. are each of them not greater than Tu.

Then $\triangle OAP - \triangle OAB = \triangle ABP = \triangle$ of (height AL) and (base AP),

$\triangle OBQ - \triangle OBC = \triangle BCQ = \triangle$ of (height BM) and (base BQ),

&c.

$\therefore$ sum of external triangles – sum of internal triangles is not greater than $\triangle$ of (height Tu) and (base $AP + BQ + \ldots\ldots$) and is $\therefore$ less than $\triangle$ of (height Tu) and (base AT), i.e. less than the assigned area. Q.E.D.

62. PROP. IV. The area of the *sector* of a circle is equal to the *triangle*, which has its base equal to the *radius* and its height equal to the *arc* of the sector.

Suppose the sector divided as above. Then

every *internal* triangle has the radius for base and an *internal* perpendicular for height: and

every *external* triangle has the radius for base and an *external* perpendicular for height.

$\therefore$ *sum of the internal* triangles is equal to a triangle having its base equal to the radius and its height equal to the *sum of the internal* perpendiculars:

and *sum of the external* triangles is equal to a triangle having its base equal to the radius and its height equal to the *sum of the external* perpendiculars.

Now the *arc* of the sector lies between the two sums of perpendiculars. (Prop. II.)

$\therefore$ the triangle having the *radius* for base and *arc* for height lies between the two sums of triangles.

But the area of the *sector* also lies between the two sums of triangles. (Prop. I. Cor.)

$\therefore$ the difference between the area of the *sector* and the triangle having the radius for base and arc for height is less than the difference between the two sums of triangles.

But this latter difference may be made less than any assignable area. (Prop. III.)

$\therefore$ *a fortiori*, no difference can be assigned between the area of the sector and the triangle having the radius for base and arc for height: i.e. these are equal. Q.E.D.

63. Prop. V. Sectors of circles which contain equal angles are to one another in the duplicate ratio of their radii.

Suppose the sector AOZ divided as above: and that another sector aoz, whose angle aoz = angle AOZ, is *similarly* divided: i.e. so that

$\angle aob = \angle AOB$, $\angle boc = \angle BOC$, &c.: and $\angle xoy = \angle XOY$.

Then, since the whole angle aoz = whole angle AOZ,

$\therefore$ the remaining angle yoz = remaining angle YOZ.

Then, since $ao : AO = bo : BO$ and $\angle aob = \angle AOB$,

$\therefore$ triangles aob, AOB are similar,

$\therefore$ $\triangle aob$ is to $\triangle AOB$ in the duplicate ratio of ao to AO.

$\therefore$ sum of the internal $\triangle$s of aoz is to sum of the internal $\triangle$s of AOZ in the duplicate ratio of ao to AO: i.e. as square on ao is to square on AO.

Similarly, sum of the external $\triangle$s of aoz is to sum of the external triangles of AOZ as square on ao is to square on AO.

But sector AOZ lies between its external and internal triangles. (Prop. I.)

$\therefore$ the fourth proportional to square on AO, square on ao, and sector AOZ lies between the external and internal triangles of aoz: and, therefore, differs from the sector aoz by a difference less than that between the external and internal triangles of aoz.

But this latter difference may be made less than any assignable area. (Prop. III.)

$\therefore$ the fourth proportional to square on AO, square on ao, and sector AOZ is equal to the sector aoz: i.e. sector AOZ : sector aoz = square on AO : square on ao, or, the sectors are in the duplicate ratio of their radii.

64. PROP. VI. The arcs of circles, which subtend equal angles, are to one another as their radii.

For, by Prop. IV., sector $AOZ = \frac{1}{2}$ rectangle OA, AZ,
and sector $aoz = \frac{1}{2}$ rectangle ao : az.

$\therefore$ by Prop. V.

$\frac{1}{2}$ rectangle OA, AZ : $\frac{1}{2}$ rectangle oa : az = sq. on OA : sq. on oa.

$\therefore$ AZ : az = OA : oa. Q.E.D.*

65. These propositions are fundamental in all questions connecting *angles* and *circles*. Thus, by Prop. VI., we prove the equality of all radians, and the constancy of the ratio of circumference to diameter.

Hence, by Prop. IV., if θ be the circular measure of an angle,

sector containing $\theta = \frac{1}{2}\theta r^2$,
and area of circle $= \pi r^2$.

EXAMPLES II.

1. Express the circumference of a circle in an integral number of inches and assign the accuracy of your result from each of the following data:—

(1) Given diameter, to within an inch, = 7 yds.

(2) Given diameter, to within an inch, = 1 mile.

(3) Given radius, to within an inch, = 8 yds. 2 ft. 3 in.

2. Reduce to circular measure the following angles:

(1) 1°. (2) 1^g. (3) $\pi°$. (4) π^g. (5) 2° 3′. (6) 15° 30′. (7) 12°.

* To aid the student in learning the above six propositions, he should note that Props. I. III. V., which concern *Sectorial areas* and not at all *Lengths of arcs* are independent of the other three: but not *vice versâ*.

3. Express in degrees the angles whose circular measures are

(1) π. (2) 3π. (3) $\frac{5\pi}{2}$. (4) $\frac{1}{3}$. (5) 3. (6) $3\frac{1}{7}$.

4. Find in degrees the angles, at the centre of a circle of radius 3 yards, subtended by the following arcs:

(1) 11 in. (2) 3 yds. 2 ft. (3) 1 ft. 10 in. (4) 11 yards.

5. The sum of two angles is $1{\cdot}4\pi$ radians. There are twice as many grades in one of them as degrees in the other. Find them.

6. Find the arcs of a circle of diameter 4 ft. which subtend at the centre the following angles:

(1) 30°. (2) 49°. (3) 70°. (4) $22\frac{1}{2}°$.

7. Find the area of a clock-face whose radius is $3\frac{1}{2}$ ft.

8. Find the area covered in 7 minutes by a minute-hand which is 1 ft. 6 in. long.

9. Show that the circumference of a circle is a little greater than 6 times the radius added to $\frac{1}{5}$ of the side of the inscribed square.

10. Which of the following pairs of angles is the greater?

(1) 76^g *or* $1{\cdot}2^c$. (2) 126° *or* $2{\cdot}3^c$.

11. If the angle subtended at the centre of a circle by the side of a regular heptagon be the unit of angular measure, by what numbers are the angles 45°, 15°, 1 right-angle expressed?

12. If the circumference of a circle be divided into five parts in arithmetical progression, the greatest part being six times the least, express in radians the angle each subtends at the centre.

CHAPTER III.

THE TRIGONOMETRICAL RATIOS AND THEIR FUNDAMENTAL RELATIONS.

[In this and the following chapters we confine ourselves to *acute* angles and to a geometrical, not trigonometrical, view of the angle.]

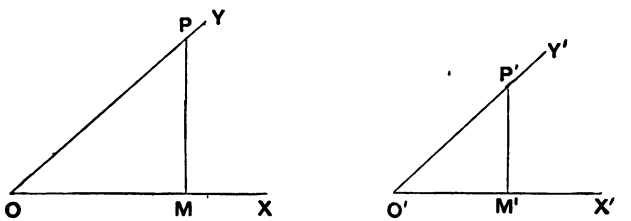
66. Let XOY be any acute angle.

In either of the lines containing it, say OX, take any point M.

Draw MP at right-angles to OX, meeting OY in P.

[Euc. Axiom XII.

Let $X'O'Y'$ be another acute angle: and let the same construction be represented by dashed letters.



67. Prop. I. If the angle XOY is equal to the angle $X'O'Y'$, then the ratio of any pair of sides of MOP shall be equal to the ratio of the corresponding pair of sides of $M'O'P'$.

For the angles at O and O', and the angles at M and M' are equal.

$\therefore$ the triangles MOP, $M'O'P'$ are equiangular. [Euc. I. 32.

$\therefore$ $OM : OP : MP = O'M' : O'P' : M'P'$. [Euc. VI. 4.

68. Prop. II. If the ratio of any one pair of sides of MOP is equal to that of the corresponding pair of sides of $M'O'P'$, then the angle XOY shall be equal to the angle $X'O'Y'$.

For the angles at M and M' are equal, being right-angles.

$\therefore$ each of the other angles is less than a right-angle.

Also the ratio of one pair of sides of MOP is equal to the ratio of the corresponding pair of sides of $M'O'P'$,

$\therefore$ the triangles MOP, $M'O'P'$ are equiangular. [Euc. VI. 6, 7.

$\therefore$ the angle XOY = the angle $X'O'Y'$.

69. The two converse propositions above proved are of fundamental importance in all that follows. Their results may be expressed thus:

The determination of the *magnitude of any acute angle* determines also the *ratio of every pair of sides* of the right-angled triangle containing that acute angle; and, conversely,

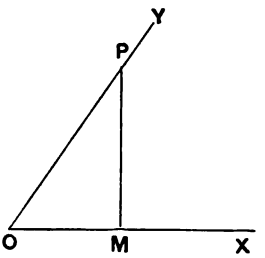
The determination of the *ratio of any one pair of sides* of a right-angled triangle determines also the *magnitude of the acute angles* of that triangle.

Briefly—the *acute angle* and its *rectangular ratios* vary together and are constant together.

70. Let XOY be any acute angle.

In either of its containing lines, say OX, take any point M.

Draw MP at right-angles to OX meeting OY in P.



Then, *in reference to the angle* XOY, *or* MOP,

(1) OM is called the **base.**

(2) OP is called the **hypothenuse** (hyp.).

(3) MP is called the **perpendicular** (perp.).

It will be convenient to observe and retain the order O, M, P; which indicates the order in which both the lines themselves and the letters which express them are written.

The *hypothenuse* is that side which is opposite to the right-angle.

The *base* and *perpendicular* are distinguished by reference to the angle in question:

i.e. the **base** is **adjacent** to the angle in question,

and the **perp.** is **opposite** to the angle in question.

Thus, *in reference to the angle MPO*,

MO, being *opposite* to the angle *MPO*, is *its* perp.:
and *PM*, being *adjacent* to the angle *MPO*, is *its* base.

71. Names have been given to the 6 ratios of the sides of the right-angled triangle as follows.

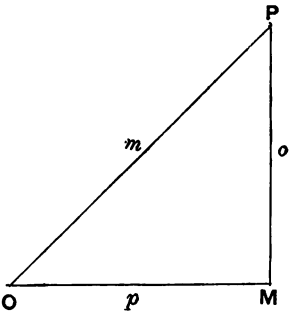
1. **Sine** $= \frac{\text{perp.}}{\text{hyp.}}$.
2. **Co-sine** $= \frac{\text{base}}{\text{hyp.}}$.
3. **Tangent** $= \frac{\text{perp.}}{\text{base}}$.
4. **Co-tangent** $= \frac{\text{base}}{\text{perp.}}$.
5. **Secant** $= \frac{\text{hyp.}}{\text{base}}$.
6. **Co-secant** $= \frac{\text{hyp.}}{\text{perp.}}$.

72. The following observations will aid the student in learning these definitions.

(1) We will call the Sine, Tangent, and Secant the *Primary Ratios*; the Cosine, Cotangent, and Cosecant the *Secondary Ratios*.

(2) Note, then, that each Secondary Ratio is derived from the corresponding Primary Ratio by interchanging *Base* and *Perpendicular*.

(3) In the triangle *OMP*, call each side by the *little* letter corresponding to the angle opposite it. Then, we follow *the same order of lettering* in the Primary Ratios as in the Sides.



Thus

Base $= OM$	Hyp. $= OP$	Perp. $= MP$
Sin. $= o : m$	Tan. $= o : p$	Sec. $= m : p$.

73. Just as the lines *base*, *hyp.*, and *perp.* are defined *in reference to a certain angle*, so the names of the ratios must be understood as defined *in reference to that angle*.

If A denotes any angle, we speak of the sine of A, the cosine of A, &c. These expressions are abbreviated into $\sin A$, $\cos A$, &c.

If A represents the angle MOP in the preceding figures, the following will be the more complete definitions of these ratios :

1. $\text{Sin } A = \dfrac{MP}{OP}$.
2. $\text{Cos } A = \dfrac{OM}{OP}$.
3. $\text{Tan } A = \dfrac{MP}{OM}$.
4. $\text{Cot } A = \dfrac{OM}{MP}$.
5. $\text{Sec } A = \dfrac{OP}{OM}$.
6. $\text{Cox } A^* = \dfrac{OP}{MP}$.

74. We have shown that for every determinate value of the acute angle A, there is a determinate value of each of the ratios of the pairs of sides of the right-angled triangle containing A; and conversely.

Hence $\sin A$, $\cos A$, &c. are what are called *Functions* of A : i.e. quantities which depend on A and vary as A varies. They are also *single-valued functions* of A, i.e. they have *one* determinate value for every determinate value of A.

75. The student must specially remember that

(1) These functions of angles, *being ratios of a line to a line*, are *abstract numbers*.

(2) Sin A is a compound symbol representing a *single numerical quantity*. It cannot be divided algebraically into *sin* and A. Thus, we cannot say $2 \sin A = \sin 2A$, or $\sin (A + B) = \sin A + \sin B$, as if *sin* were a multiplier or factor. We

* Usually written *cosec A*. In the abbreviations of the *secondary ratios*, the student should attend to the *final* letters *s*, *t*, *x*.

must treat $\sin A$ as one whole symbol, as if it were always enclosed in an impregnable bracket. $(\sin A)^2$, $(\sin A)^3$, &c., are, however, always written in the forms $\sin^2 A$, $\sin^3 A$, &c., in order that the bracket may be dispensed with in writing.

So of the other ratios.

76. *Explanation of the prefix* Co.

In the triangle OMP, since the angle OMP is a right-angle,

$\therefore$ the angles MOP, MPO are together equal to a right-angle,

$\therefore$ the angles MOP, MPO are *Complementary*.

Now the side OM, which is *base* to $\angle MOP$, is *perp.* to $\angle MPO$, and the side MP, which is *perp.* to $\angle MOP$, is *base* to $\angle MPO$.

$\therefore$ the ratios of MPO will be obtained from those of MOP by *interchanging base and perp.* But this is equivalent to an interchange between the *corresponding primary and secondary ratios.*

Hence,

$$\begin{aligned} \text{cosin } A &= \text{sin of } \textit{com}\text{plement of } A, \\ \text{cotan } A &= \text{tan of } \textit{com}\text{plement of } A, \\ \text{cosec } A &= \text{sec of } \textit{com}\text{plement of } A. \end{aligned}$$

Since, similarly,

$$\begin{aligned} \sin A &= \text{cosin of } \textit{com}\text{plement of } A, \\ \tan A &= \text{cotan of } \textit{com}\text{plement of } A, \\ \sec A &= \text{cosec of } \textit{com}\text{plement of } A, \end{aligned}$$

we may say

$$\sin = \text{co-cosin}, \quad \tan = \text{co-cotan}, \quad \sec = \text{co-cosec}.$$

The above results should be carefully worked out by a reference to the figure.

77. Since, by above, the *cosine* of any angle is the *sine* of some other angle, and so of the others, therefore, *general* propositions about the *sine* will be true of the *cosine*; and those about the *tangent* will be true of the *cotangent*; and those about the *secant* will be true of the *cosecant*. The next article will illustrate this statement.

Fundamental relations of Inequality *between the ratios of any one angle.*

78. From the proposition that *of the three sides of a right-angled triangle the hypothenuse is larger than either of the others*, the following inequalities amongst the ratios of an angle may be deduced:

Since, *sine* and *cosine* have hypothenuse in *denominator*, **sine** and **cosine** are always **less** than unity.

Since *secant* and *cosecant* have hypothenuse in *numerator*, **secant** and **cosecant** are always **greater** than unity.

Since *sine* and *cosine* have hypothenuse in *denominator*, and have the same numerator as *tangent* and *cotangent* respectively, **sine** is **less** than **tangent**, and **cosine** is **less** than **cotangent**.

Since *secant* and *cosecant* have hypothenuse in *numerator*, and the same denominator as *tangent* and *cotangent* respectively, **secant** is **greater** than **tangent**, and **cosecant** than **cotangent**.

Thus the following trios are in ascending order:

sine, tangent, secant: cosine, cotangent, cosecant.

Fundamental relations of Equality *between the ratios of any one angle.*

79. *The dual or reciprocal relations.*

(1) $\sin A \times \operatorname{cox} A = \frac{\text{perp.}}{\text{hyp.}} \times \frac{\text{hyp.}}{\text{perp.}} = 1.$

(2) $\tan A \times \cot A = \frac{\text{perp.}}{\text{base}} \times \frac{\text{base}}{\text{perp.}} = 1.$

(3) $\sec A \times \cos A = \frac{\text{hyp.}}{\text{base}} \times \frac{\text{base}}{\text{hyp.}} = 1.$

The triple or zigzag relations.

(4) $\sin A \times \cot A \times \sec A = \frac{\text{perp.}}{\text{hyp.}} \times \frac{\text{base}}{\text{perp.}} \times \frac{\text{hyp.}}{\text{base}} = 1.$

(5) $\cos A \times \tan A \times \operatorname{cox} A = \frac{\text{base}}{\text{hyp.}} \times \frac{\text{perp.}}{\text{base}} \times \frac{\text{hyp.}}{\text{perp.}} = 1.$

There are thus 3 independent dual relations, and 2 independent triple relations. These 5 relations are not all independent. *Any one of them may be deduced algebraically from the other four.*

These equations follow at once from the *definitions* of the ratios, and do not at all depend on any property of the right-angled triangle.

80. *The Squared or Pythagorean relations.*

By the famous 47th Proposition of Euc. I., attributed by tradition to Pythagoras, we have, in any right-angled triangle,

sum of squares on sides containing the right angle
= square on side opposite the right angle.

Thus, algebraically,

$$(\text{Perp.})^2 + (\text{Base})^2 = (\text{Hyp.})^2.$$

To reduce this equation to ratio form, we may divide successively by $(\text{hyp.})^2$, $(\text{base})^2$, and $(\text{perp.})^2$.

Thus

$$(6)\quad \left(\frac{\text{Perp.}}{\text{Hyp.}}\right)^2 + \left(\frac{\text{Base}}{\text{Hyp.}}\right)^2 = \left(\frac{\text{Hyp.}}{\text{Hyp.}}\right)^2, \text{ i.e. } \sin^2 A + \cos^2 A = 1.$$

$$(7)\quad \left(\frac{\text{Perp.}}{\text{Base}}\right)^2 + \left(\frac{\text{Base}}{\text{Base}}\right)^2 = \left(\frac{\text{Hyp.}}{\text{Base}}\right)^2, \text{ i.e. } \tan^2 A + 1 = \sec^2 A.$$

$$(8)\quad \left(\frac{\text{Perp.}}{\text{Perp.}}\right)^2 + \left(\frac{\text{Base}}{\text{Perp.}}\right)^2 = \left(\frac{\text{Hyp.}}{\text{Perp.}}\right)^2, \text{ i.e. } 1 + \cot^2 A = \text{cox}^2 A.$$

There are thus 3 independent squared relations. Of the 8 relations given in this and the preceding article, any 5, including *one* at least of the squared relations, are independent; *from these* 5 *the other* 3 *may be algebraically deduced.*

These squared relations depend on the Pythagorean property of the right-angled triangle, being in fact merely transformed expressions of that property.

81. The relations of *inequality* of Art. 78, will help the student to remember the relations of *equality* of Arts. 79 and 80.

Thus, the equation $sin^2 A + cos^2 A = 1$, which may be written

$$\sin^2 A = 1 - \cos^2 A \text{ or } \cos^2 A = 1 - \sin^2 A,$$

shows that *sin A* and *cos A* are each less than 1.

The equation $\tan^2 A + 1 = \sec^2 A$, which may be written

$$\tan^2 A = \sec^2 A - 1 \text{ or } \sec^2 A - \tan^2 A = 1,$$

shows that *sec A* is greater than 1 and greater than *tan A*.

The equation $1 + \cot^2 A = \text{cox}^2 A$, which may be written

$$\cot^2 A = \text{cox}^2 A - 1 \quad or \quad \text{cox}^2 A - \cot^2 A = 1,$$

shows that *cox A* is greater than 1 and greater than *cot A*.

The equation *sin A* × *cox A* = 1, which may be written

$$\sin A = \frac{1}{\text{cox } A} \quad or \quad \text{cox } A = \frac{1}{\sin A},$$

shows that of *sin A* and *cox A* one is less and the other greater than 1.

The equation *sec A* × *cos A* = 1, which may be written

$$\sec A = \frac{1}{\cos A} \quad or \quad \cos A = \frac{1}{\sec A},$$

shows that of *sec A* and *cos A* one is greater and the other less than 1.

82. *Definition.* Two quantities whose product is unity are called **Reciprocals.**

The reciprocal of unity is unity.

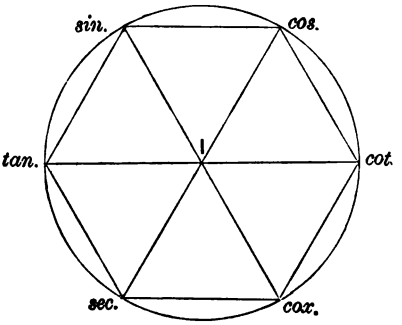
Of two reciprocals, not equal to 1, one must be greater and the other less than 1.

Thus, sin and cox, tan and cot, sec and cos, $\frac{2}{3}$ and $\frac{3}{2}$, 5 and $\frac{1}{5}$, are pairs of *reciprocal* quantities.

Observe that *tan* and *cot* form a pair which is both *complementary* and *reciprocal.*

83. The above and any other relations between the trigonometrical ratios may be conveniently studied and remembered by means of

The Ratio-Hexagon.



Arrange the ratios, in the order of their definitions, symmetrically on the circumference of a circle, so that the corresponding ratios which are complementary to one another are in the same horizontal line: and place *unity* at the centre.

84. The relations of Art. 78, are summed up thus:—

On either side the quantities increase as we descend.

85. The relations of Art. 79, are summed up thus:—

The product of any set of ratios arranged symmetrically about the centre is equal to unity.

86. The relations of Art. 80, are summed up thus:—

The sum of the squares of any two horizontally adjacent quantities is equal to the square of the quantity just below and between them.

87. As an exercise, the student may show also that (1) each ratio equals the product of its two neighbours. E.g. $\tan = \sin \times \sec$. (2) Each ratio equals the ratio of its neighbour to the next on the same side. E.g. $\tan = \sin \div \cos$.

88. The most important formulæ for use in Examples, &c., are the following 5 which are independent. These should be learnt by everyone.

dual. (1) $\operatorname{cox} A = \dfrac{1}{\sin A}$, (2) $\sec A = \dfrac{1}{\cos A}$,

triple. (3) $\tan A = \dfrac{\sin A}{\cos A}$, (4) $\cot A = \dfrac{\cos A}{\sin A}$,

squared. (5) $\sin^2 A + \cos^2 A = 1$.

Example 1. Prove each of the above identities immediately from the definitions of the ratios.

Method. Substitute in the more complex sides of each identity,

(1) $\dfrac{1}{\sin A} = 1 \div \sin A = 1 \div \dfrac{MP}{OP} = \dfrac{OP}{MP} = \operatorname{cox} A$,

(2) $\dfrac{1}{\cos A} = 1 \div \cos A = 1 \div \dfrac{OM}{OP} = \dfrac{OP}{OM} = \sec A$,

(3) $\dfrac{\sin A}{\cos A} = \sin A \div \cos A = \dfrac{MP}{OP} \div \dfrac{OM}{OP} = \dfrac{MP}{OP} \times \dfrac{OP}{OM} = \dfrac{MP}{OM} = \tan A$,

$$(4)\quad \frac{\cos A}{\sin A} = \cos A \div \sin A = \frac{OM}{OP} \div \frac{MP}{OP} = \frac{OM}{OP} \times \frac{OP}{MP} = \frac{OM}{MP} = \cot A,$$

$$(5)\quad \sin^2 A + \cos^2 A = \frac{MP^2}{OP^2} + \frac{OM^2}{OP^2} = \frac{MP^2 + OM^2}{OP^2} = (\text{Euc. I. 47})\ \frac{OP^2}{OP^2} = 1.$$

Example 2. Show from definitions that $\tan A + \cot A = \sec A \,.\, \operatorname{cox} A$.

$$\tan A + \cot A = \frac{MP}{OM} + \frac{OM}{MP} = \frac{(MP)^2 + (OM)^2}{OM \,.\, MP}$$

$$= (\text{Euc. I. 47})\ \frac{OP^2}{OM \,.\, MP} = \frac{OP}{OM} \cdot \frac{OP}{MP}$$

$$= \sec A \,.\, \operatorname{cox} A.$$

89. The 5 formulæ of the last article may be used in proving any given complex formula. Thus, by means of the first four formulæ, *any other ratio may be expressed in terms of* **sine** *and* **cosine**. By means of the fifth, *either of the two ratios* **sine** *and* **cosine** *may be expressed in terms of the other.*

This method will prove any required formulæ. But it is not always the shortest method. The student should by degrees familiarise himself with other formulæ, especially the two—

$$\sec^2 A = 1 + \tan^2 A \text{ and } \operatorname{cox}^2 A = 1 + \cot^2 A.$$

90. *Example* 1. Prove the identity

$$\sec^2 A + \operatorname{cox}^2 A = \sec^2 A \,.\, \operatorname{cox}^2 A.$$

By (1) and (2) of art. 88

$$\sec^2 A + \operatorname{cox}^2 A = \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A} = \frac{\sin^2 A + \cos^2 A}{\sin^2 A \,.\, \cos^2 A}$$

$$= \text{by (5)}\ \frac{1}{\sin^2 A \,.\, \cos^2 A}.$$

Again, by (1) and (2) of art. 88

$$\sec^2 A \,.\, \operatorname{cox}^2 A = \frac{1}{\cos^2 A} \cdot \frac{1}{\sin^2 A} = \frac{1}{\cos^2 A \,.\, \sin^2 A},$$

$$\therefore\ \sec^2 A + \operatorname{cox}^2 A = \sec^2 A \,.\, \operatorname{cox}^2 A. \qquad \text{Q. E. D.}$$

Example 2. Prove that

$$\tan^2 A - \cot^2 A = \sec^2 A - \operatorname{cosec}^2 A.$$

$$\tan^2 A - \cot^2 A = \frac{\sin^2 A}{\cos^2 A} - \frac{\cos^2 A}{\sin^2 A} = \frac{\sin^4 A - \cos^4 A}{\cos^2 A \,.\, \sin^2 A}$$

$$= \frac{(\sin^2 A + \cos^2 A)(\sin^2 A - \cos^2 A)}{\cos^2 A \,.\, \sin^2 A} = \frac{\sin^2 A - \cos^2 A}{\cos^2 A \,.\, \sin^2 A}$$

[because $\sin^2 A + \cos^2 A = 1$].

Again,

$$\sec^2 A - \operatorname{cosec}^2 A = \frac{1}{\cos^2 A} - \frac{1}{\sin^2 A} = \frac{\sin^2 A - \cos^2 A}{\cos^2 A \,.\, \sin^2 A}.$$

Example 3. Prove that

$$\frac{\sec A + 1}{\sec A - 1} - \frac{\sec A - 1}{\sec A + 1} = 4 \cot A \operatorname{cosec} A.$$

Left side of above

$$= \frac{(\sec A + 1)^2 - (\sec A - 1)^2}{\sec^2 A - 1}$$

$$= \frac{(\sec^2 A + 2 \sec A + 1) - (\sec^2 A - 2 \sec A + 1)}{\sec^2 A - 1} = \frac{4 \sec A}{\sec^2 A - 1} = \frac{4 \sec A}{\tan^2 A}$$

$$= \frac{4}{\cos A} \div \frac{\sin^2 A}{\cos^2 A} = \frac{4}{\cos A} \times \frac{\cos^2 A}{\sin^2 A} = \frac{4 \cos A}{\sin^2 A} = 4 \frac{\cos A}{\sin A} \cdot \frac{1}{\sin A}$$

$$= 4 \cot A \,.\, \operatorname{cosec} A.$$

Example 4. Prove that

$$\frac{1 + \tan A}{\sec A} = \frac{\sec A + 2 \sin A}{1 + \tan A},$$

i.e. *that* $(1 + \tan A)^2 = \sec A (\sec A + 2 \sin A)$,

i.e. *that* $1 + 2 \tan A + \tan^2 A = \sec^2 A + 2 \sin A \sec A$,

i.e. *that* $1 + \tan^2 A + 2 \tan A = \sec^2 A + 2 \tan A$,

which is clearly true.

91. We have shown that the *cosine* and *sine* are always *less* than unity. Hence

Def. The defect of the *cosine* from unity is called the **versed sine**; and the defect of the *sine* from unity is called the **coversed sine.**

Or briefly,

$$1 - \cos A = \operatorname{vers} A \text{ and } 1 - \sin A = \operatorname{covers} A.$$

Again, we have shown that the *secant* and *cosecant* are always *greater* than unity. Hence we might adopt the following definitions:—

Def. The excess of the *secant* over unity is called the **subtense**; and the excess of the *cosecant* over unity is called the **cosubtense.**

Or briefly

$$\sec A - 1 = \text{subt } A \text{ and } \operatorname{cosec} A - 1 = \text{cosubt } A.$$

92. *Example.* If from the hypothenuse a length equal to the base is cut off, show that the ratio of the remainder to the hypothenuse is the *versed sine*, and the ratio of the remainder to the base is the *subtense.*

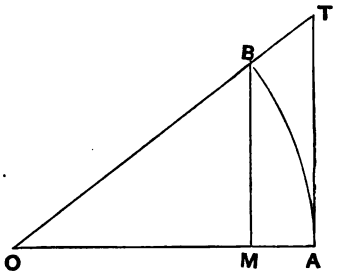
93. The following explanation may be given of the origin of the terms used in this chapter.

Let AB be a portion of the circumference of a circle whose centre is O.

Draw the radii OA, OB.

From A draw AT, perpendicular to OA, touching the circle at A and meeting OB produced in T.

From B draw BM perpendicular to OA.



Then the circumference AB, being likened to a *bow*, was called the *arc* [L. arcus]; the perpendicular MB, where the bow-string touches the *bosom*, was called the *sine* [L. sinus]; the line AT, *touching* the circle, was called the *tangent* [L. tangens]; the line OT, *cutting* the circle, was called the *secant* [L. secans]; the line AM, over against the sine MB, was called the *versed sine*; and the line BT, which *subtends* the angle BAT contained by the chord and tangent, is sometimes called the *subtense.*

Thus, in the older trigonometry, the *sine*, *tangent*, *secant*, &c., were functions of the *arc* AB, not of the angle AOB. Moreover they represented certain *lengths*, not certain ratios. They, therefore, varied, not only as the angle AOB varied, but also as the radius OA varied.

94. The connection between the ancient and modern nomenclature is thus indicated :—

If we divide each *linear function of the arc* by the *radius of the circle*, we obtain the corresponding *ratio-function of the angle*. Thus

$$\frac{\text{linear sine}}{\text{radius}} = \frac{MB}{OB} = \text{sine of angle } AOB.$$

$$\frac{\text{linear tangent}}{\text{radius}} = \frac{AT}{OA} = \text{tangent of angle } AOB.$$

$$\frac{\text{linear secant}}{\text{radius}} = \frac{OT}{OA} = \text{secant of angle } AOB.$$

$$\frac{\text{linear versed sine}}{\text{radius}} = \frac{MA}{OA} = \text{versed sine of angle } AOB.$$

$$\frac{\text{linear subtense}}{\text{radius}} = \frac{BT}{OB} = \text{subtense of angle } AOB.$$

95. Now, as the angle AOB or the arc AB increases, it is clear that each of the lines MB, AT, OT, AM, BT increases. Hence the *primary* ratios sine, tangent, secant, versed sine, subtense *increase as the angle increases.*

The functions, having the prefix *co-*, are found from the above by substituting for the arc or angle its *complement*. Hence the *secondary* ratios *decrease as the angle increases.*

96. Any relations which hold amongst the *linear functions* will involve the *radius* of the circle. The corresponding relations amongst the ratio-functions may be found from these by putting *radius* $= 1$. Conversely, we may find the relations amongst the linear functions, from those involving the ratio-functions, by introducing *radius* wherever it is necessary to make the equations homogeneous. If we place *rad.* instead of 1 in the centre of the ratio-hexagon, we shall see the symmetry of the relations more clearly than before. Thus, connecting the linear functions, we have

$$\sin^2+\cos^2=\text{rad}^2, \qquad \tan^2+\text{rad}^2=\sec^2, \qquad \text{rad}^2+\cot^2=\text{cox}^2$$

$$\sin . \sec = \text{rad} . \tan \text{ \&c.}, \qquad \cos . \sec = \text{rad}^2 \text{ \&c.}$$

$$\sin^2 . \tan^2+\sin^2 . \text{rad}^2=\tan^2 . \text{rad}^2,$$

$$\cos^2 . \text{rad}^2+\cos^2 . \cot^2=\cot^2 . \text{rad}^2,$$

$$\text{rad}^2 . \sec^2+\text{rad}^2 . \text{cox}^2=\sec^2 . \text{cox}^2.$$

Examples III.

1. Let ABC be a triangle, right-angled at C, such that $AB = 6$ ft., $BC = \sqrt{11}$ ft. Find the trigonometrical ratios of the angles at A and B.

2. The sides of a right-angled triangle are in the ratio $3 : 4 : 5$. Find sine, cosine, and tangent, of each of the acute angles.

3. The sides of a right-angled triangle are in the ratio $2 : 3\sqrt{5} : 7$. Find secant, cosecant, and cotangent, of each of acute angles.

4. Write down five independent formulæ by means of which the other ratios may be expressed immediately in terms of $\sec A$ and $\tan A$, and these in terms of one another.

5. Write down similar formulæ for $\operatorname{cosec} A$ and $\cot A$.

6. Why is $\sin 30° = \cos 60°$?

7. Show that $\sin 45° = \cos 45°$, and thence find $\tan 45°$.

8. Apply the equation $\sin^2 A + \cos^2 A = 1$, to find $\sin 45°$.

9. Find the angle whose cosine is equal to $\sin 70°$.

10. In a triangle, right-angled at C, the sides opposite A, B, and C, are called a, b, and c, respectively.

Prove (i) $a \sin B = b \sin A$.

(ii) $2a^2 \tan B + b^2 \sec^2 A = c^2(\sin A + \sin B)^2$.

11. Prove

(i) $\sin^2 A = \dfrac{\tan^2 A}{1 + \tan^2 A}$, (ii) $\cos^2 A = \dfrac{\cot^2 A}{1 + \cot^2 A}$,

(iii) $\sec^2 A = \dfrac{1 + \cot^2 A}{\cot^2 A}$, (iv) $\operatorname{cosec}^2 A = \dfrac{1 + \tan^2 A}{\tan^2 A}$.

12. Express

(i) $(1 + \sin^2 \theta)^2$ in terms of $\cos \theta$.

(ii) $(1 - \tan^2 \theta)^2$ in terms of $\sec \theta$.

13. Show from definition and figure

(i) $\operatorname{cosec} A - \sin A = \cos A \,.\, \cot A.$

(ii) $\sec A - \cos A = \sin A \,.\, \tan A.$

(iii) $\sec^2 A + \operatorname{cosec}^2 A = \sec^2 A \,.\, \operatorname{cosec}^2 A.$

(iv) $\tan^2 A - \sin^2 A = \tan^2 A \,.\, \sin^2 A.$

(v) $\cot^2 A - \cos^2 A = \cot^2 A \,.\, \cos^2 A.$

Prove the following identities:

14. $\dfrac{\sec A - 1}{\tan A} = \dfrac{\tan A}{\sec A + 1}.$

15. $\dfrac{1 + \tan A}{1 - \tan A} = \dfrac{\cot A + 1}{\cot A - 1}.$

16. $1 + \tan\theta \cos^2\theta + \cot\theta \sin^2\theta = (\sin\theta + \cos\theta)^2.$

17. $\dfrac{1 + \cos\theta}{1 - \cos\theta} - \dfrac{1 - \cos\theta}{1 + \cos\theta} = 4 \cot\theta \,.\, \operatorname{cosec}\theta.$

18. $\sqrt{\dfrac{1 - \sin A}{1 + \sin A}} = \sec A - \tan A.$

19. $\dfrac{\sin A}{1 + \cos A} = \dfrac{1 - \cos A}{\sin A}.$

20. $\dfrac{1 - \cos A}{1 + \cos A} = (\operatorname{cosec} A - \cot A)^2.$

21. $\sec A + \sin^2 A + \cot^2 A = \operatorname{cosec}^2 A + \cos^2 A \,(\sec^3 A - 1).$

22. $\sec^3\theta = \sec\theta + \tan^3\theta \,.\, \operatorname{cosec}\theta.$

23. $2 \sec^2\theta = \dfrac{1}{1 - \sin\theta} + \dfrac{1}{1 + \sin\theta}.$

24. $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} + \dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$
$= \dfrac{\sec\theta \operatorname{cosec}\theta}{2} \left(\dfrac{\operatorname{cosec}\theta + \sec\theta}{\operatorname{cosec}\theta - \sec\theta} - \dfrac{\operatorname{cosec}\theta - \sec\theta}{\operatorname{cosec}\theta + \sec\theta} \right).$

25. $(\sec A \,.\, \cot A + 1)(\sec A \,.\, \cot A - 1) = \cos^2 A \,.\, \operatorname{cosec}^2 A.$

26. $(\sec^2 A + \tan^2 A)(\operatorname{cosec}^2 A + \cot^2 A) = 1 + 2 \sec^2 A \,.\, \operatorname{cosec}^2 A.$

27. $\cot^4 A + \cot^2 A = \operatorname{cosec}^4 A - \operatorname{cosec}^2 A.$

28. $\sec^4 A - \sec^2 A = \tan^4 A + \tan^2 A.$

29. $(\sec^2 A \cdot \operatorname{cosec}^2 A - \tan^2 A)(\sec^2 A \cdot \operatorname{cosec}^2 A - \cot^2 A)$
$= 1 + 2 \sec^2 A \operatorname{cosec}^2 A.$

30. $(\cos^2 A + \cot^2 A) \tan^2 A = \sec^2 A + (\cos^2 A - 1) \tan^2 A.$

31. $\cos^4 A - \sin^4 A = \cos^2 A - \sin^2 A.$

32. $\cos^4 A + \sin^4 A = 1 - 2 \sin^2 A \cdot \cos^2 A.$

33. $\cos^6 A + \sin^6 A = 1 - 3 \sin^2 A \cdot \cos^2 A.$

34. $\cos^8 A - \sin^8 A = (\cos^2 A - \sin^2 A)(1 - 2 \sin^2 A \cdot \cos^2 A).$

35. $\sec^6 \theta = 1 + \tan^6 \theta + 3 \tan^2 \theta \cdot \sec^2 \theta.$

36. $\operatorname{vers}^2 A - 2 \operatorname{vers} A + \sin^2 A = 0.$

37. $\tan A - \sin A = \tan A \operatorname{vers} A.$

38. $\operatorname{vers} A = \cos A \operatorname{subt} A.$

39. $\operatorname{vers} A (1 + \sec A) = \operatorname{subt} A (1 + \cos A).$

40. $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A.$

41. $\tan^2 A + \sec^2 B = \sec^2 A + \tan^2 B.$

42. $\dfrac{\cos A + \cos B}{\sec A + \sec B} = \cos A \cdot \cos B = \dfrac{\cos B - \cos A}{\sec A - \sec B}.$

43. $(\sin A + \sin B)(\operatorname{cosec} B - \operatorname{cosec} A) = (\sin A - \sin B)$
$(\operatorname{cosec} B + \operatorname{cosec} A).$

44. $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \dfrac{\cos B - \cos A}{\sin A - \sin B}.$

45. $(\sin A - \cos B)(\sin A + \cos B) = (\sin B - \cos A)$
$(\sin B + \cos A).$

46. Solve for $\sin \theta$:—

(i) $3 \sin^2 \theta - \cos^2 \theta = 6 \cot^2 \theta.$

(ii) $\sin \theta + \cos \theta = 1.$

(iii) $\tan \theta + \sec \theta = 2.$

(iv) $\cot \theta + \operatorname{cosec} \theta = 3.$

(v) $4 \cos^2 \theta + 2 \sin \theta = 4 + \sqrt{3}(1 - 2 \sin \theta).$

47. Solve for $\cos \theta$:—$\cot \theta + \operatorname{cosec} \theta = 5.$

48. Solve for $\tan \theta$:—$12 \sec^2 \theta = 6 + 17 \tan \theta.$

49. In the ratio-hexagon, the ratio of the *product* to the *sum* of the squares of any two horizontally adjacent quantities is equal to the square of the quantity just above and between them.

CHAPTER IV.

RATIOS OF PARTICULAR ANGLES.

§ 1. One Ratio being given.

97. *Given the value of any one ratio of an angle, to draw the angle and to find the values of the other ratios.*

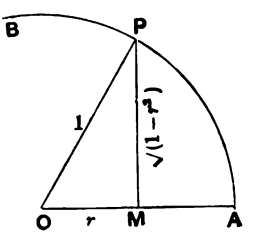
General Method. Let $p : q$ be the given value of the ratio. Construct a right-angled triangle, in which the measures of those *two* sides, by which the ratio is defined, are respectively p and q. Then find the *third* side by Euc. I. 47.

98. *Given the value of the* **sine** *or* **cosine** *of any angle, to draw the angle.*

Let the given value be r, i.e. $r : 1$.

Then, since *sin* and *cos* are always less than 1, r is less than 1.

Take a line OA equal to any unit length. With centre O and radius OA describe a circle APB. From OA cut off a part OM equal to r times OA.



Draw MP at right-angles to OA, cutting the circle in P. Join OP.

Then the measure of $OP = OA$ is 1.

And the measure of $OM = r \,.\, OA$ is r.

$$\therefore \cos MOP = \sin MPO = \frac{OM}{OP} = \frac{r}{1} = r.$$

Therefore an angle has been drawn whose *sine* or whose *cosine* has the given value r. Q.E.F.

If r had been greater than 1, OM would have been greater than OA, M would have been outside the circle, *and* MP *would not have cut the circle at all.* Hence the method would have failed as we should expect.

99. *Given the value of the* **sine** *or* **cosine** *of any angle, to find the other ratios.*

In other words: To express the other ratios in terms of the *sine* or *cosine*.

As before, let r be the value of the given ratio.

Construct the figure of the last article.

Then, by Euc. I. 47, $MP^2 + OM^2 = OP^2$,

$$\text{i.e. } MP^2 + r^2 = 1,$$

$$\therefore MP^2 = 1 - r^2,$$

$$\therefore MP = \sqrt{(1 - r^2)}.$$

(i) Suppose that r is the *sine* of the required angle. Then MPO is the required angle, which we may call A.

$$\therefore r = \sin A.$$

$$\therefore \cos A = \cos MPO = \frac{PM}{PO} = \frac{\sqrt{(1-r^2)}}{1} = \sqrt{(1 - \sin^2 A)},$$

$$\tan A = \tan MPO = \frac{MO}{PM} = \frac{r}{\sqrt{(1-r^2)}} = \frac{\sin A}{\sqrt{(1 - \sin^2 A)}},$$

and so for the other ratios.

(ii) Suppose that r is the *cosine* of the required angle. Then MOP is the required angle, which we may call A.

$$\therefore r = \cos A.$$

$$\therefore \sin A = \sin MOP = \frac{MP}{OP} = \frac{\sqrt{(1-r^2)}}{1} = \sqrt{(1 - \cos^2 A)},$$

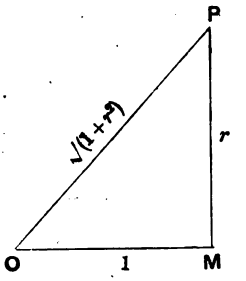
$$\tan A = \tan MOP = \frac{MP}{OM} = \frac{\sqrt{(1-r^2)}}{r} = \frac{\sqrt{(1 - \cos^2 A)}}{\cos A},$$

and so for the other ratios.

100. *Given the value of the* **tangent** *or* **cotangent** *of any angle, to draw the angle.*

Let the given value be r, i.e. $r : 1$.

Take $OM = 1$: draw MP at right-angles to OM, and let $MP = r$. Join OP.



Then $\tan MOP = \cot MPO$

$$= \frac{MP}{OM} = \frac{r}{1} = r.$$

Therefore an angle has been drawn whose *tangent* or whose *cotangent* has the given value r. Q. E. F.

101. *Given the value of the* **tangent** *or* **cotangent** *of any angle, to find the other ratios.*

In other words: To express the other ratios in terms of the *tangent* or *cotangent*.

From the figure of the last article,

by Euc. I. 47, $OP^2 = OM^2 + MP^2 = 1 + r^2$,

$$\therefore\ OP = \sqrt{(1 + r^2)}.$$

(i) Suppose $r = tan\ A$ so that $MOP = A$.

Then, $\sin A = \sin MOP = \dfrac{MP}{OP} = \dfrac{r}{\sqrt{(1+r^2)}} = \dfrac{\tan A}{\sqrt{(1+\tan^2 A)}}$,

and so for the other ratios.

(ii) Suppose $r = cot\ A$, so that $MPO = A$.

Then, $\sin A = \sin MPO = \dfrac{MO}{PO} = \dfrac{1}{\sqrt{(1+r^2)}} = \dfrac{1}{\sqrt{(1+\cot^2 A)}}$,

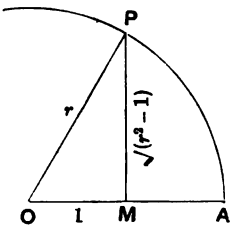
and so for the other ratios.

102. *Given the value of the* **secant** *or* **cosecant** *of any angle, to draw the angle.*

Let the given value be r, i.e. $r : 1$.

Then r must be greater than 1.

Take $OA = r$; describe a circle with centre O and radius OA. From OA cut off $OM = 1$. Draw MP at right angles to OA meeting the circle in P.



Then $OP = OA = r$.

$$\therefore \sec MOP = \operatorname{cosec} MPO = \frac{OP}{OM} = \frac{r}{1} = r.$$

Therefore an angle has been drawn whose *secant* or whose *cosecant* has the given value r. Q. E. F.

103. *Given the value of the* **secant** *or* **cosecant** *of any angle, to find the other ratios.*

In other words: To express the other ratios in terms of the *secant* or *cosecant*.

From the figure of the last article, by Euc. I. 47,

$$MP^2 = OP^2 - OM^2 = r^2 - 1,$$

$$\therefore \quad MP = \sqrt{(r^2 - 1)}.$$

(i) Suppose $r = sec\ A$, so that $MOP = A$.

Then, $$\sin A = \sin MOP = \frac{MP}{OP} = \frac{\sqrt{(r^2-1)}}{r} = \frac{\sqrt{(\sec^2 A - 1)}}{\sec A},$$

and so for the other ratios.

(ii) Suppose $r = cosec\ A$, so that $MPO = A$.

Then, $$\sin A = \sin MPO = \frac{OM}{OP} = \frac{1}{r} = \frac{1}{\operatorname{cosec} A},$$

and so for the other ratios.

104. The results of Arts. 99, 101, 103, may also be obtained, without drawing a figure, from the general relations of equality established in the preceding chapter.

Example 1. To express the *tangent* in terms of the *cosecant*.

$$\tan A = \frac{1}{\cot A} = \frac{1}{\sqrt{(\operatorname{cosec}^2 A - 1)}}.$$

Example 2. To express the *sine* in terms of the *tangent*.

$$\sin A = \tan A \times \cos A = \frac{\tan A}{\sec A} = \frac{\tan A}{\sqrt{(1+\tan^2 A)}}.$$

105. Other problems may be solved in a somewhat similar way. It is left to the student to prove the results of the following examples.

Example 1. To divide any angle AOB into two angles whose *secants* shall have a given ratio.

Cut off OA and OB, so that $OA : OB =$ given ratio.

Join AB: and draw OC perpendicular to AB.

OC shall divide AOB (internally or externally) as required.

Example 2. To divide any angle AOB into two angles whose *tangents* shall have a given ratio.

Let $CD : DE =$ given ratio. Place CD and DE in a straight line. Upon CE describe a segment CFE of a circle containing an angle equal to AOB. Draw DF at right-angles to CE cutting the circumference in F.

DF shall divide CFE, which is equal to AOB, as required.

Example 3. To divide any angle AOB into two angles whose *sines* shall have a given ratio.

Produce AO to A': and cut off OA' and OB so that $OB : OA' =$ given ratio. Draw OC parallel to $A'B$.

OC shall divide AOB as required.

§ 2. The Angle being given.

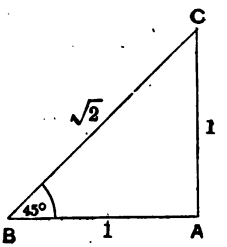
106. We now proceed to find the ratios of such angles as have simple geometrical relations by a method similar to the above. It is convenient to take the *smallest* line in our figure as the *unit of length*, in order to avoid unnecessary fractions. A second side is then found by some proposition connected with the particular angle in question. Lastly, a third side is always found by Euc. I. 47.

The student should in each case retain the figure with the values of its *sides and angles* marked, and not try at first to remember the *ratios themselves*.

107. *The ratios of the* half *right-angle.* $\left[45°, 50^g, \frac{\pi^c}{4}.\right]$

Draw AC, AB equal and at right-angles to one another. Join BC.

Or briefly: *Construct a right-angled isosceles triangle.*



Then since $AB = AC$, $\therefore \angle ABC = \angle ACB$; [Euc. I. 5.

but these angles are together one right-angle. [Euc. I. 32.

$\therefore$ each of them is a *half* right-angle.

Take $AB = AC$ for unit length. Then

$$BC^2 = AB^2 + AC^2 = 1 + 1 = 2, \qquad \text{[Euc. I. 47.}$$

$$\therefore BC = \sqrt{2},$$

$$\therefore \sin 45° = \frac{AC}{BC} = \frac{1}{\sqrt{2}} : \cos 45° = \frac{BA}{BC} = \frac{1}{\sqrt{2}},$$

$$\tan 45° = \frac{AC}{BA} = 1 \quad : \cot 45° = \frac{BA}{AC} = 1,$$

$$\sec 45° = \frac{BC}{BA} = \sqrt{2} : \operatorname{cox} 45° = \frac{BC}{AC} = \sqrt{2}.$$

Since 45° is its own complement, any ratio of 45° is the same as its *co*-ratio.

108. *The ratios of* $\frac{1}{3}$ *and* $\frac{2}{3}$ *of a right-angle.*

$$\left[30° \text{ and } 60° : \frac{\pi^c}{6} \text{ and } \frac{\pi^c}{3}.\right]$$

On AB as base describe an equilateral triangle ABC, and bisect the angle ACB by CD.

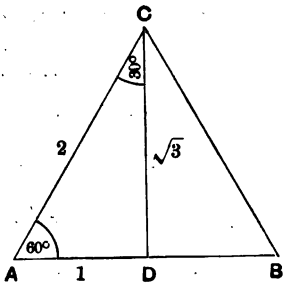
Or briefly: *Construct a bisected equilateral triangle.*

Then ABC being equilateral is also equiangular. [Euc. I. 5.

$\therefore$ each of its angles $= \frac{1}{3}$ of 2 right-angles $= 60°$, [Euc. I. 32.

$\therefore$ the angle $ACD = \frac{1}{3}$ of a right-angle $= 30°$,

and the angle $DAC = \frac{1}{3}$ of 2 right-angles $= 60°$.



Also, as in Euc. I. 10, the triangles DAC, DBC are equal in all respects.

$\therefore$ $AD = DB$; and DC is at right-angles to AB.

Take AD for unit length.

Then $AC = AB = 2 \,.\, AD = 2.$

And $DC^2 + AD^2 = AC^2.$ [Euc. I. 47.

i.e. $DC^2 + 1 = 2^2 = 4,$

$\therefore$ $DC^2 = 3$ and $DC = \sqrt{3}.$

Thus, $\sin 60° = \frac{DC}{AC} = \frac{\sqrt{3}}{2}$: $\cos 60° = \frac{AD}{AC} = \frac{1}{2}$: &c.,

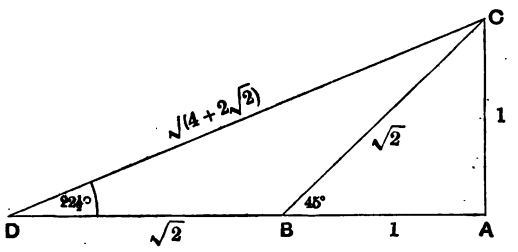
and $\sin 30° = \frac{AD}{AC} = \frac{1}{2}$: $\cos 30° = \frac{DC}{AC} = \frac{\sqrt{3}}{2}$: &c.

Since 30° and 60° are *complementary*, any ratio of the one is equal to the *co*-ratio of the other.

109. *The ratios of* $\frac{1}{4}$ *and* $\frac{3}{4}$ *of a right-angle.*

$$\left[22\tfrac{1}{2}° \text{ and } 67\tfrac{1}{2}° : \frac{\pi^c}{8} \text{ and } \frac{3\pi^c}{8}.\right]$$

Describe the isosceles triangle ABC, right-angled at A. Produce AB to D, making $BD = BC$. Join DC.



Then, as in Art. 107, $\angle ABC$ is a $\frac{1}{2}$ right-angle; and the angles BDC, BCD being equal by Euc. I. 5, each of them is $\frac{1}{4}$ of a right-angle by Euc. I. 32.

Take $AB = AC = 1$. Then $BD = BC = \sqrt{2}$, as in Art. 107.

$$\therefore \cot 22\tfrac{1}{2}^\circ = \cot ADC = \frac{DA}{AC} = \sqrt{2} + 1.$$

Again $DC^2 = DA^2 + AC^2 = (2 + 2\sqrt{2} + 1) + 1 = 4 + 2\sqrt{2}$.

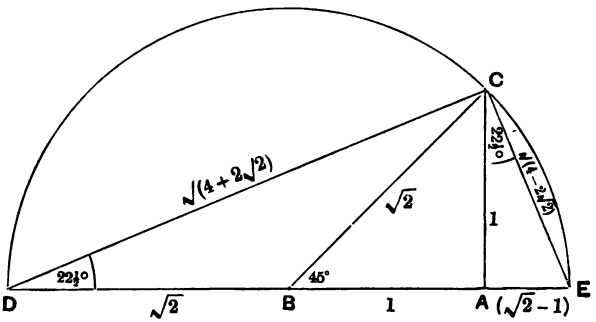
$$\therefore \operatorname{cosec} 22\tfrac{1}{2}^\circ = \operatorname{cosec} ADC = \frac{DC}{AC} = \sqrt{(4 + 2\sqrt{2})}.$$

Similarly the other ratios may be written down, and their denominators may be rationalised by an algebraic process.

The same figure gives the ratios of $67\frac{1}{2}^\circ$.

By the following simple geometrical construction; the other ratios may be at once written down with rationalised denominators.

With centre B and radius BD or BC, describe the semi-circle DCE, cutting BA produced in E. Join CE.



Then $\angle DCE$ in a semi-circle is a right-angle.

$\therefore \angle ACE =$ complement of $\angle DCA = \angle ADC = 22\frac{1}{2}^\circ$.

Also $AE = BE - BA = \sqrt{2} - 1$,

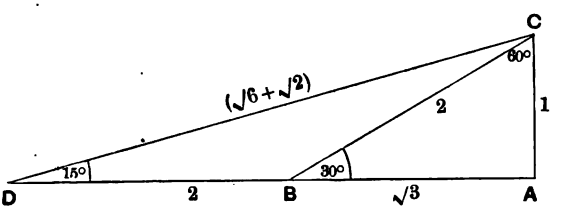
$\therefore CE^2 = AE^2 + AC^2 = (2 - 2\sqrt{2} + 1) + 1 = 4 - 2\sqrt{2}$,

$\therefore$ from $\triangle EDC$, $\sin 22\frac{1}{2}^\circ = \frac{1}{2}\sqrt{(2 - \sqrt{2})}$ *and* $\cos 22\frac{1}{2}^\circ = \frac{1}{2}\sqrt{(2 + \sqrt{2})}$,

and from $\triangle ECA$, $\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1$ *and* $\sec 22\frac{1}{2}^\circ = \sqrt{(4 - 2\sqrt{2})}$.

110. *The ratios of $\frac{1}{6}$ and $\frac{5}{6}$ of a right-angle.*

$$\left[15^\circ \text{ and } 75^\circ : \frac{\pi^c}{12} \text{ and } \frac{5\pi^c}{12}.\right]$$



Let ABC be the half of an equilateral triangle, so that, as in Art. 108, $\angle BCA = 60°$, $\angle ABC = 30°$, $\angle BAC = 90°$. Produce AB to D, making $BD = BC$. Join DC.

Then, the angles BDC, BCD being equal, [Euc. I. 5.
each of them is half angle ABC, i.e. 15°. [Euc. I. 32.

Take $AC = 1$. Then $BD = BC = 2$, and $AB = \sqrt{3}$, as in Art. 108.

$$\therefore \cot 15° = \cot ADC = \frac{DA}{AC} = 2 + \sqrt{3}.$$

Again,

$$DC^2 = DA^2 + AC^2 = (4 + 4\sqrt{3} + 3) + 1 = 8 + 4\sqrt{3} = 6 + 2\sqrt{12} + 2,$$

$$\therefore DC = \sqrt{6} + \sqrt{2}.$$

$$\therefore \operatorname{cosec} 15° = \operatorname{cosec} ADC = \frac{DC}{AC} = \sqrt{6} + \sqrt{2}.$$

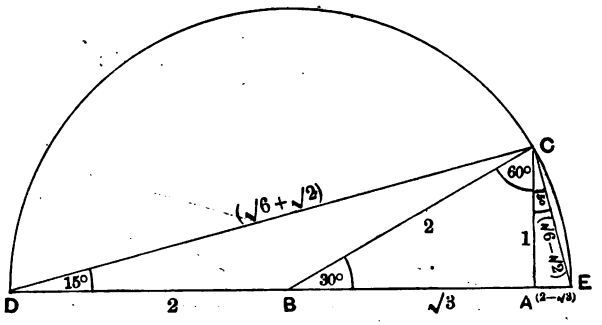
Similarly the other ratios of 15° and 75° may be written down, and their denominators may be rationalised by an algebraic process.

By the same construction as in the preceding article, the other ratios may be at once written down with rationalised denominators.

With centre B and radius BD or BC, describe the semi-circle DCE, cutting BA produced in E. Join CE.

Then $\angle DCE$ in a semi-circle is a right-angle,

$\therefore$ $\angle ACE$ = complement of $\angle DCA$ = $\angle ADC = 15°$.



Also $$AE = BE - BA = 2 - \sqrt{3},$$

$$\therefore\ CE^2 = AE^2 + AC^2 = (4 - 4\sqrt{3} + 3) + 1 = 8 - 4\sqrt{3},$$

$\therefore$ from $\triangle EDC$, $\sin 15° = \frac{1}{4}(\sqrt{6} - \sqrt{2})$ *and* $\cos 15° = \frac{1}{4}(\sqrt{6} + \sqrt{2})$,

and from $\triangle ECA$, $\tan 15° = 2 - \sqrt{3}$ *and* $\sec 15° = \sqrt{6} - \sqrt{2}$.

111. We may proceed to bisect the angles $22\frac{1}{2}°$ and $15°$ in the same way. The student will easily show that

$$\cot 11\tfrac{1}{4}° = \sqrt{2} + 1 + \sqrt{(4 + 2\sqrt{2})},$$

$$\cot 7\tfrac{1}{2}° = 2 + \sqrt{3} + \sqrt{6} + \sqrt{2}.$$

In precisely the same way the student may prove geometrically the general formula

$$\cot A = \operatorname{cosec} 2A + \cot 2A.$$

112. *The ratios of* $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$ *of a right angle.*

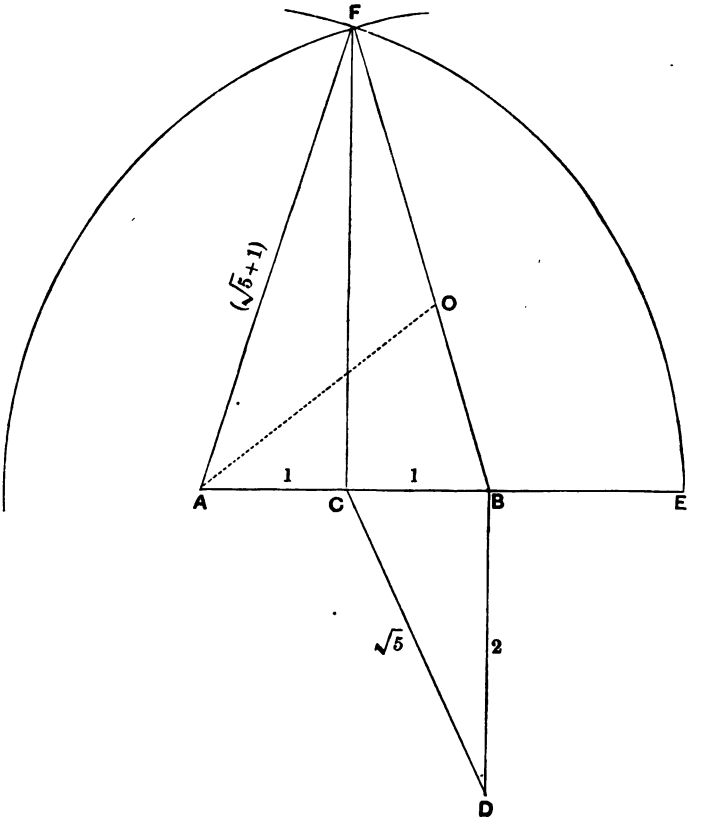
[18°, 36°, 54°, 72°.]

Bisect any straight line AB in C. Draw BD at right angles and equal to AB. Join CD. Produce AB to E, making CE equal to CD. With centres A and B and radii each equal to AE describe two circles cutting in F. Join FA, FB, so that FA and FB are each equal to AE. *The angle* AFB *shall be* 36°.

For $\because$ AB is bisected in C and produced to E,

$$\therefore\ AE \cdot BE + BC^2 = CE^2 = CD^2 = BD^2 + BC^2,$$

$$\therefore\ AE \cdot BE = BD^2 \text{ or } AB^2.$$



From FB, which is equal to AE, cut off FO equal to AB; then the remainder BO = remainder BE.

$\therefore$ the above equation becomes $BF \cdot BO = AB^2$,

$$\therefore BF : AB = AB : BO,$$

$\therefore$ the triangles FBA, ABO, having the sides about their common angle B proportional, are similar.

$\therefore \angle AOB = \angle FAB = \angle FBA$ *and* $AO = AB = FO$.

But $\because AO = FO$, $\therefore \angle AFO = \angle FAO$, $\therefore \angle AOB =$ *twice* $\angle AFO$.

$\therefore$ each of the angles FAB, FBA of the triangle FAB is double of the third angle AFB.

$$\therefore \angle AFB + 2 . \angle AFB + 2 . \angle AFB = 180°, \text{ i.e. } 5 . \angle AFB = 180°,$$

$$\therefore \angle AFB = 36°.$$

Join FC. Then FC bisects angle AFB and is perpendicular to AB.

$$\therefore \angle AFC = 18°, \text{ and cosec } AFC = \frac{AF}{AC}.$$

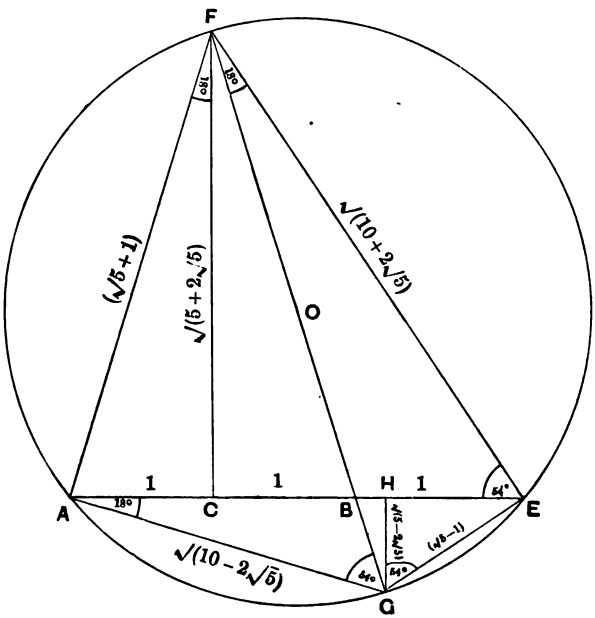
Now *let* $AC = CB = 1$. *Then* $AB = BD = 2$,

and $CE^2 = CD^2 = CB^2 + BD^2 = 1 + 2^2 = 5$, $\therefore CE = \sqrt{5}$.

and $AF = BF = AE = CE + AC = \sqrt{5} + 1$.

$$\therefore \text{ cosec } 18° = \sqrt{5} + 1.$$

The following construction gives the ratios of 18° or 72° and of 54° or 36° in their simplest form.



In the above figure, with centre O and radius OF or OA describe a circle $FAGE$ cutting AB produced in E, and FB produced in G. From F and G drop FC, GH perpendiculars on AE. These perpendiculars being drawn from the extremities F, G of the diameter FG cut off equal segments from the chord AE.

$$\therefore\ HE=AC \text{ and } AH=CE.$$

Consider the *angles* of the figure.

$\because\ FAG=90°$ *and* $FAB=72°$, $\therefore\ BAG=18°$,

$\therefore\ GFE=GAE=18°\ .\ AEG=AFG=36°\ .\ FGE=FAE=72°$,

$\because\ FEG=90°$ *and* $AEG=36°$, $\therefore\ AEF=54°=AFE$, $\therefore\ AE=AF$.

And $GBE=ABF=72°=BGE$, $\therefore\ GE=BE$.

Consider the *lengths* in the figure.

Let $AC=HE=1$. Then, by above, $AE=AF=\sqrt{5}+1$.

And $CE=AH=\sqrt{5}$, $\therefore\ GE=BE=CE-CB=\sqrt{5}-1$,

$\therefore\ CF^2=AF^2-AC^2=5+2\sqrt{5}$ *and* $GH^2=GE^2-HE^2=5-2\sqrt{5}$,

$\therefore\ EF^2=CF^2+CE^2=10+2\sqrt{5}$ *and* $AG^2=GH^2+AH^2=10-2\sqrt{5}$.

Lastly, $FG=2\ .\ FO=2\ .\ AB=4$,

$\therefore$ from $\triangle GFE$, $\sin 18°=\frac{1}{4}(\sqrt{5}-1)$, $\cos 18°=\frac{1}{4}\sqrt{(10+2\sqrt{5})}$,

from $\triangle FGA$, $\sin 54°=\frac{1}{4}(\sqrt{5}+1)$, $\cos 54°=\frac{1}{4}\sqrt{(10-2\sqrt{5})}$,

from $\triangle GAH$, $\tan 18°=\dfrac{\sqrt{(5-2\sqrt{5})}}{\sqrt{5}}$, $\sec 18°=\dfrac{\sqrt{(10-2\sqrt{5})}}{\sqrt{5}}$,

from $\triangle FEC$, $\tan 54°=\dfrac{\sqrt{(5+2\sqrt{5})}}{\sqrt{5}}$, $\sec 54°=\dfrac{\sqrt{(10+2\sqrt{5})}}{\sqrt{5}}$,

from $\triangle AFC$, $\cot 18°=\sqrt{(5+2\sqrt{5})}$, $\cos 18°=\sqrt{5}+1$,

from $\triangle EGH$, $\cot 54°=\sqrt{(5-2\sqrt{5})}$, $\cos 54°=\sqrt{5}-1$.

It may be observed that the lengths *below* AE have the same form as the corresponding lengths *above* AE, but a *minus* sign instead of a *plus* sign. Thus the corresponding ratios of 18° and of 54° have the same algebraical form, except for the sign of the second term. Moreover, the $-$ sign occurs in the *primary* ratios of 18° and in the *secondary* ratios of 54°. This last fact may be connected with the results of Art. 114.

Example. Hence show that

$$\cot 9°=\sqrt{(5+2\sqrt{5})}+\sqrt{5}+1, \text{ and } \cot 27°=\sqrt{(5-2\sqrt{5})}+\sqrt{5}-1.$$

113. *To find the limiting values of the ratios as the triangle of reference collapses into a straight line.*

The triangle of reference OMP collapses into a straight line,

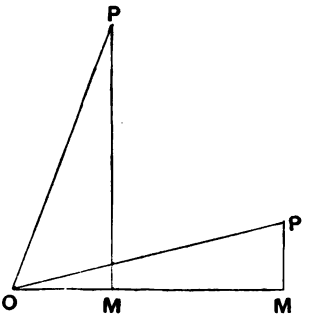
when either the base OM or the perp. MP coincides with the hyp. OP, while the remaining side vanishes. Thus,

as MOP decreases to 0°,

OM becomes equal to OP, and MP vanishes;

as MOP increases to 90°,

MP becomes equal to OP, and OM vanishes.



Now the ratio of two finite equal lengths is *unity*: $\frac{f}{f}=1$.

The ratio of zero to any finite length is *zero*: $\frac{0}{f}=0$.

The ratio of any finite length to zero is *infinity*: $\frac{f}{0}=\infty$.

[For a fraction varies in the *same* direction as its *numerator*, but in the *opposite* direction to its *denominator*. Hence, if the numerator decreases indefinitely, the fraction decreases indefinitely :—$0/f=0$. But if the denominator decreases indefinitely, the fraction increases indefinitely :—$f/0=\infty$.]

Hence the limiting values of the ratios are

0, 1 or ∞.

$$\sin 0=\frac{0}{f}=0:\ \tan 0=\frac{0}{f}=0:\ \sec 0=\frac{f}{f}=1:$$

$$\cos 0=\frac{f}{f}=1:\ \cot 0=\frac{f}{0}=\infty:\ \operatorname{cox} 0=\frac{f}{0}=\infty:$$

$$\sin 90°=\frac{f}{f}=1:\ \tan 90°=\frac{f}{0}=\infty:\ \sec 90°=\frac{f}{0}=\infty:$$

$$\cos 90°=\frac{0}{f}=0:\ \cot 90°=\frac{0}{f}=0:\ \operatorname{cox} 90°=\frac{f}{f}=1.$$

In this table the student should observe:

(1) That, the angles 0 and 90° being complementary, any ratio of the one = the co-ratio of the other.

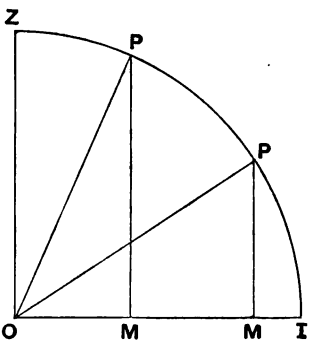
(2) Since, for certain limiting angles, the sin, cos, sec or cox equals 1, strictly we must say of sin and cos—not that they are *always less* than 1—but that they are *never greater* than 1: and of sec and cox—not that they are *always greater* than 1—but that they are *never less* than 1.

(3) The *primary* ratios have their *minimum* values at 0, and their *maximum* values at 90°.

The *secondary* ratios have their *maximum* values at 0, and their *minimum* values at 90°.

114. *General changes in value of the ratios, as the angle changes from* 0 *to* 90°.

Let a line of constant length *OP* revolve from any position *OI*. It will describe a continuously increasing angle, as its extremity *P* traces out the quadrant of a circle.



Drop a perpendicular *PM* upon *OI*.

Then, as the angle *IOP* increases from 0 to 90°,

OP remains constant: *MP* increases: *OM* decreases.

Now, the perp. *MP* is in the numerator of *sin* and *tan*;
but in the denominator of *cox* and *cot*.

And the base *OM* is in the numerator of *cos* and *cot*;
but in the denominator of *sec* and *tan*.

Thus as the angle increases from 0 to 90°, the *primary* ratios, sin, tan, sec increase: while the *secondary* ratios, cos, cot, cox decrease.

From 0 to 90°,

sin increases from 0 to 1: but *cos* decreases from 1 to 0.

tan increases from 0 to ∞: but *cot* decreases from ∞ to 0.

sec increases from 1 to ∞: but *cox* decreases from ∞ to 1.

The results of some of the preceding articles exemplify these statements: e.g.

$$\sin 0° = \frac{\sqrt{0}}{2} : \sin 30° = \frac{\sqrt{1}}{2} : \sin 45° = \frac{\sqrt{2}}{2} : \sin 60° = \frac{\sqrt{3}}{2} : \sin 90° = \frac{\sqrt{4}}{2}.$$

§ 3. The Solution of Trigonometrical Equations.

115. A Trigonometrical equation is an equation connecting the ratios of some unknown angle which has to be found.

The solution generally requires the following three steps:

(1) Express the different ratios which occur *in terms of one single ratio*; by means of the formulæ of this or the preceding chapter.

(2) Find the value of this one ratio by the purely *algebraic process* of solving an equation.

(3) Put down the acute angle, whose ratio has the value found, by applying inversely the results of this chapter on the values of the ratios of specific angles.

Example 1. Solve $2\sin\theta = 1$.

(1) Here only *one* ratio occurs.

(2) The algebraical process gives $\sin\theta = \frac{1}{2}$.

(3) Reference to Art. 108 shows that $\theta = 30°$.

Example 2. Solve $\sqrt{3}(\tan\theta + \cot\theta) = 4$.

(1) Since $\cot\theta = \dfrac{1}{\tan\theta}$ $\therefore \sqrt{3}\left(\tan\theta + \dfrac{1}{\tan\theta}\right) = 4.$

(2) $$\therefore \sqrt{3}\,.\tan^2\theta + \sqrt{3} = 4\tan\theta,$$

$$\therefore \tan^2\theta - \frac{4}{\sqrt{3}}\tan\theta = -1,$$

$$\therefore \tan^2\theta - \frac{4}{\sqrt{3}}\tan\theta + \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{4}{3} - 1 = \frac{1}{3},$$

$$\therefore \tan\theta - \frac{2}{\sqrt{3}} = \pm\frac{1}{\sqrt{3}},$$

$$\therefore \tan\theta = \frac{3}{\sqrt{3}} \text{ or } \frac{1}{\sqrt{3}} = \sqrt{3} \text{ or } \frac{1}{\sqrt{3}}.$$

(3) Reference to Art. 108 shows that $\theta = 60°$ or $30°$.

Example 3. Solve $\tan^2\theta+\sec\theta=5$.

(1) Since $\tan^2\theta=\sec^2\theta-1$, $\therefore$ $\sec^2\theta-1+\sec\theta=5$.

(2) $\therefore$ $\sec^2\theta+\sec\theta=6$,

$$\therefore \sec^2\theta+\sec\theta+\left(\frac{1}{2}\right)^2=6+\frac{1}{4}=\frac{25}{4},$$

$$\therefore \sec\theta+\frac{1}{2}=\pm\frac{5}{2},$$

$$\therefore \sec\theta=\frac{4}{2} \text{ or } -\frac{6}{2}=2 \text{ or } -3.$$

(3) Reference to Art. 108 shows that $\sec\theta=2$ is satisfied by $\theta=60°$. But negative values of the ratios have not yet been treated. Hence for the present we may discard the solution $\sec\theta=-3$.

Example 4. Solve $2\sin A=\sqrt{3}\tan A$.

Method by factors. Here

$$2\sin A-\sqrt{3}\tan A=0 \text{ gives } \sin A(2-\sqrt{3}\sec A)=0,$$

$$\therefore \text{either } \sin A=0 \text{ or } 2-\sqrt{3}\sec A=0.$$

Now $\sin A=0$ *gives* $A=0$.

And $2-\sqrt{3}\sec A=0$ *gives* $\sec A=\frac{2}{\sqrt{3}}$, $\therefore$ $A=30°$.

$$\therefore A=0 \text{ or } 30°.$$

EXAMPLES IV.

1. Draw the angle A and find all its ratios in each of the following cases:—

(1) $\sin A=\frac{5}{13}$, (2) $\tan A=3$, (3) $\sec A=1\cdot4$,

(4) $\cos A=\cdot6$, (5) $\cot A=\cdot3$, (6) $\operatorname{cosec} A=3\frac{4}{7}$.

2. Express *tan A* in terms of *sin A* and of *cos A*.

3. Express *sec A* in terms of *sin A* and of *cot A*.

4. Express *cosec A* in terms of *sec A* and of *cos A*.

5. Express *sin A* in terms of *tan A* and of *sec A*.

6. Divide an angle into two angles whose *cosines* shall have a given ratio.

7. Divide an angle into two angles whose *cosecants* shall have a given ratio.

Find the acute angles including 0 *and* 90° *which satisfy each of the following equations* (8—25).

8. $4 + 4\cos\theta = 3\sec\theta$.
9. $2\sin\theta + 2\operatorname{cosec}\theta = 5$.
10. $\tan\theta = \cot\theta$.
11. $2\sin\theta = \operatorname{cosec}\theta$.
12. $\sec^3\theta + 3\operatorname{cosec}^2\theta = 8$.
13. $2\sin\theta = \tan\theta$.
14. $\tan\theta + \cot\theta = 2$.
15. $\sqrt{2}\,.\cos\theta - \cot\theta = 0$.
16. $2\sin^2\theta = 3\cos\theta$.
17. $4\sin\theta = \operatorname{cosec}\theta - 2\sqrt{2}$.
18. $3\tan^2\theta + 2 = \cot^2\theta$.
19. $2\cos^2\theta = 2\sin^2\theta + 1$.
20. $4\sin\theta = \operatorname{cosec}\theta - 2$.
21. $\tan\theta + 4\cos^2\theta = 3$.
22. $\sec^2\theta = 2 + 2\tan\theta$.
23. $2\sin\theta\tan\theta + 1 = \tan\theta + 2\sin\theta$.
24. $2\cos^2\theta + \sqrt{3}\,.\sin\theta = 2$.
25. $\tan\theta + \cot\theta = 4$.

Prove the following identities: (26—32).

26. $(\sin 30° + \sin 60°)(\cos 30° - \cos 60°) = \sin 30°$.
27. $(\cos 60° + \cos 30°)(\sin 60° - \sin 30°) = \cos 60°$.
28. $\sin 30° + \sin 60° = \sqrt{2}\,.\cos 15°$.
29. $\cos 30° - \cos 60° = \sqrt{2}\,.\sin 15°$.
30. $(\sin 60° - \sin 45°)(\cos 45° + \cos 30°) = \sin^2 30°$.
31. $\cos 36° - \sin 18° = \frac{1}{2}$.
32. $4\sin 18°\cos 36° = 1$.

33. If the *sine* of an angle be $> \dfrac{1}{\sqrt{2}}$ and the *cosine* of that angle be $> \frac{1}{2}$, between what limits does the angle lie?

34. If the *tangent* of an angle be $> \dfrac{1}{\sqrt{3}}$ and the *cotangent* of that angle be $> \sqrt{3}$, between what limits does the angle lie?

35. Trace the changes in value of (1) $1 - \cos\theta$; (2) $\sec\theta - 1$; (3) $\sin\theta + \cos\theta$; (4) $\sec\theta - \tan\theta$: as θ changes from 0 to 90°.

Solve the following simultaneous equations:

36. $\sin^4\alpha = \cos^2\beta$, *and* $4\cot^2\alpha = \sec^2\beta$.

37. $\sin(A + B) = \dfrac{1}{\sqrt{2}}$, *and* $\cos(A - B) = \dfrac{\sqrt{3}}{2}$.

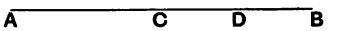
38. In Example I. 19, show that the angles ODC and DOC are, respectively, 60° and 30°.

CHAPTER V.

RATIOS OF COMPOUND ANGLES.

116. The following two propositions will be found useful in many applications.

I. *If a magnitude AB be bisected at C and divided at D,*

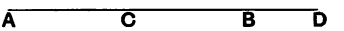


then (1) $AD + DB = 2 \,.\, AC$, and (2) $AD - DB = 2 \,.\, CD$.

For (1) $AD + DB = AB = 2 \,.\, AC$.

and (2) $AD - DB = (AC + CD) - (CB - CD) = 2 \,.\, CD$.

II. *If a magnitude AB be bisected at C and extended to D,*



then (1) $AD - BD = 2 \,.\, AC$, and (2) $AD + BD = 2 \,.\, CD$.

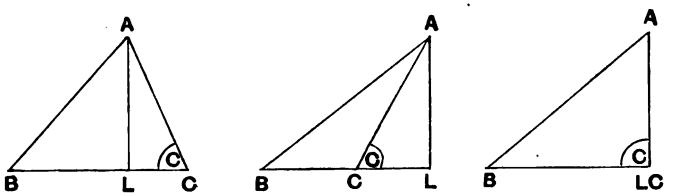
For (1) $AD - BD = AB = 2 \,.\, AC$.

and (2) $AD + BD = (AC + CD) + (CD - CB) = 2 \,.\, CD$.

These propositions may be applied to straight lines, angles, triangles, &c., as in Arts. which follow.

117. *To express the area of a triangle, when two sides and the included angle are given.*

We have shown (Art. 22) that the *area* of a triangle is measured by $\frac{1}{2}$ (*base* × *altitude*).



Let ABC be any triangle. Draw AL perpendicular to the base BC produced if necessary.

Then $\triangle ABC = \frac{1}{2} BC \,.\, LA$. (1)

The angle ACB is either acute, obtuse, or right.

In each case, let C denote that angle at the point C, *which is not obtuse.*

Then, in each case, $\sin C = \dfrac{LA}{CA}$.

$$\therefore CA \,.\, \sin C = LA.$$

Substituting for LA in (1)

$$\triangle ABC = \tfrac{1}{2}\, BC \,.\, CA \,.\, \sin C. \qquad (2)$$

Hence, *the area of a triangle is measured by half the product of two sides and the sine of the (non-obtuse) angle contained by them.*

118. *To express the sine of the internal or external angle of a triangle in terms of the area of the triangle and the sides containing the angle.*

By the last article we have, if C be the non-obtuse angle at C, which is either internal or external to the triangle ABC,

$$\tfrac{1}{2}\, BC \,.\, CA \sin C = \triangle ABC,$$

$$\therefore \sin C = \frac{2 \,.\, \triangle ABC}{BC \,.\, CA}.$$

Hence, *the sine of the internal or external angle of a triangle*

$$= \frac{\text{twice area of the triangle}}{\text{product of sides containing the angle}}.$$

The student should observe that both the numerator and the denominator of the above fraction represent *an area.* Hence the sine is given correctly as an *abstract* number.

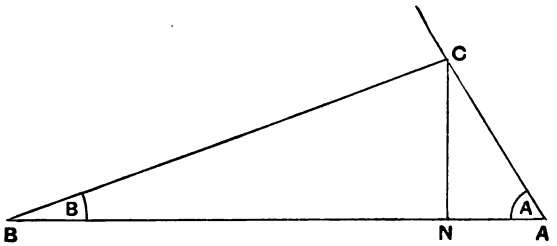
119. We proceed to find the ratios of angles formed by the addition or subtraction of two angles.

We shall adopt a method which in each case involves the application of the preceding article. But alternative proofs are given in Chapter XIII., which the student may use in preference to these.

The beginner is very liable to fall into the mistake of supposing that *sin* $(A+B)$ must equal *sin* A + *sin* B, that *sin* $2A$ must equal 2 *sin* A, and so on. He must remember, as was pointed out before, that *sin* is not a *multiplier* which can be separated from its accompanying symbols. Thus, *sin* $(A+B)$ means "The sine of the angle formed by *adding the two angles* A and B"; while *sin* A + *sin* B means "The number formed by *adding the two sines* of the angles."

In what follows all the angles whose ratios are involved are ordinary (i.e. positive) acute angles.

120. *To prove the formula* $\sin(A+B)=\sin A\cos B+\cos A\sin B$..(1).



Let A, B be the angles at the base of a triangle, whose vertical angle at C is *obtuse*. Draw CN perpendicular to AB.

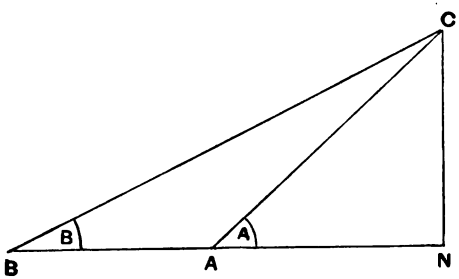
Then, producing AC, the exterior angle at C is *acute* and interior $\angle A$ + interior $\angle B$ = exterior $\angle C$. [Euc. I. 32.

$$\therefore \sin(A+B)=\sin(\text{exterior } \angle C)=\frac{2\triangle ABC}{AC\,.\,BC}=\frac{NC\,.\,BA}{AC\,.\,BC}$$

$$=\frac{NC(BN+NA)}{AC\,.\,BC}=\frac{NC}{AC}\cdot\frac{BN}{BC}+\frac{NA}{AC}\cdot\frac{NC}{BC}$$

$$=\sin A\,.\,\cos B+\cos A\,.\,\sin B.$$

121. *To prove the formula* $\sin(A-B)=\sin A\cos B-\cos A\sin B$..(2).



Let A be an (acute) angle exterior, and B an angle interior, to the triangle ABC. Draw CN perpendicular to BA produced.

Then exterior $\angle A$ − interior $\angle B$ = interior $\angle C$. [Euc. I. 32.

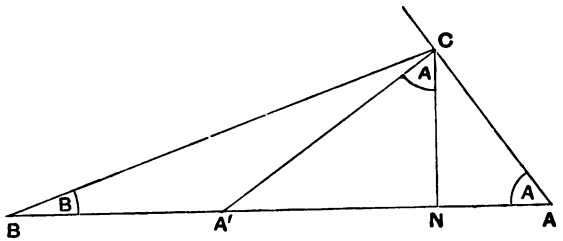
$$\therefore \sin(A-B)=\sin(\text{interior } \angle C)=\frac{2\triangle ABC}{AC\,.\,BC}=\frac{NC\,.\,BA}{AC\,.\,BC}$$

$$=\frac{NC(BN-AN)}{AC\,.\,BC}=\frac{NC}{AC}\cdot\frac{BN}{BC}-\frac{AN}{AC}\cdot\frac{NC}{BC}$$

$$=\sin A\,.\,\cos B-\cos A\,.\,\sin B.$$

122. *To prove the formula*

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \ldots\ldots\ldots(3).$$



In the figure of Art. 120, draw CA' at right angles to CA.

Then $\angle NCA' = \text{comp. of } \angle NCA = \angle NAC = A$,

and int. $\angle BCA' = \text{comp. of ext. } \angle C = \text{comp. of } (A + B)$.

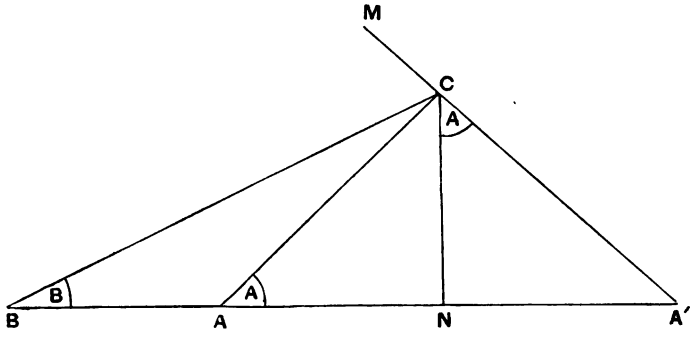
$$\therefore \cos(A + B) = \sin BCA' = \frac{2\triangle A'BC}{A'C \cdot BC} = \frac{NC \cdot BA'}{A'C \cdot BC}$$

$$= \frac{NC \cdot (BN - A'N)}{A'C \cdot BC} = \frac{NC}{A'C} \cdot \frac{BN}{BC} - \frac{A'N}{A'C} \cdot \frac{NC}{BC}$$

$$= \cos A \cdot \cos B - \sin A \cdot \sin B.$$

123. *To prove the formula*

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \ldots\ldots\ldots(4).$$



In the figure of Art. 121, draw MCA' at right angles to CA.

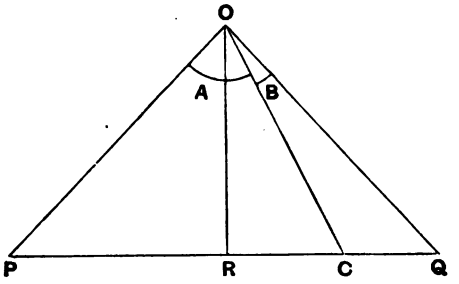
Then $\angle NCA' = \text{comp. of } \angle NCA = \angle NAC = A$,

and ext. $\angle BCM = \text{comp. of int. } \angle C = \text{comp. of } (A - B)$.

$$\therefore \cos(A-B) = \sin BCM = \frac{2\triangle A'BC}{A'C \,.\, BC} = \frac{NC \,.\, BA'}{A'C \,.\, BC}$$
$$= \frac{NC\,(BN+NA')}{A'C \,.\, BC} = \frac{NC}{A'C}\cdot\frac{BN}{BC} + \frac{NA'}{A'C}\cdot\frac{NC}{BC}$$
$$= \cos A \,.\, \cos B + \sin A \,.\, \sin B.$$

124. *To prove the formulæ*

$$\sin A + \sin B = 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) \ldots\ldots (5),$$
$$\sin A - \sin B = 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \ldots\ldots (6).$$



Let $\angle POC = A$; and $\angle QOC = B$. Take $OP = OQ$. Join PQ, cutting OC in C.

Let OR bisect angle POQ, triangle POQ, and base PQ.

Then (Art. 116, I.)

$\triangle POC + \triangle COQ = 2\triangle POR$ *and* $\triangle POC - \triangle COQ = 2\triangle ROC$.

$\angle POC + \angle COQ = 2\angle POR$ *and* $\angle POC - \angle COQ = 2\angle ROC$.

$$\therefore \sin A + \sin B = \frac{2\triangle POC}{OP \,.\, OC} + \frac{2\triangle COQ}{OQ \,.\, OC} = \frac{4\triangle POR}{OP \,.\, OC}$$
$$= \frac{2PR}{OP}\cdot\frac{OR}{OC}$$
$$= 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B),$$

$$\text{and } \sin A - \sin B = \frac{2\triangle POC}{OP \,.\, OC} - \frac{2\triangle COQ}{OQ \,.\, OC} = \frac{4\triangle ROC}{OP \,.\, OC}$$
$$= \frac{2OR}{OP}\cdot\frac{RC}{OC}$$
$$= 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B).$$

Similarly we may show that

(17) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$. (18) $\sec 2A = \dfrac{\sec^2 A}{2-\sec^2 A}$.

(19) $\cos 2A = 2\cos^2 A - 1$. (20) $\cos 2A = 1 - 2\sin^2 A$.

127. *To find the tangent and cotangent of the sum or difference of two angles.*

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}.$$

[Arts. 120, 122.

In order to express this fraction in terms of *tan A* and *tan B*, we must divide its numerator and denominator by *cos A . cos B*.

$$\therefore \tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B} \quad \ldots\ldots\ldots\ldots (21).$$

Similarly: $$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B} \quad \ldots\ldots\ldots\ldots (22),$$

$$\cot(A+B) = \frac{\cot A\cot B - 1}{\cot B + \cot A} \quad \ldots\ldots\ldots\ldots (23),$$

$$\cot(A-B) = \frac{\cot A\cot B + 1}{\cot B - \cot A} \quad \ldots\ldots\ldots\ldots (24).$$

128. The preceding 24 formulæ are not independent of one another. It will be useful to show their algebraical interdependence. This is done in Arts. 129—132.

129. *Given* $\sin(A+B) = \sin A\cos B + \cos A\sin B$ .....(1),

to prove $\cos(A-B) = \cos A\cos B + \sin A\sin B$ .....(4).

In (1) let $A = 90° - C$.

Then $\sin A = \sin(90° - C) = \cos C$,

and $\cos A = \cos(90° - C) = \sin C$,

and $\sin(A+B) = \sin(90° - C - B) = \cos(C - B)$.

$\therefore$ (1) *becomes* $\cos(C-B) = \cos C\cos B + \sin C\sin B$.

Writing A for C we have (4). Similarly

Given $\sin(A-B) = \sin A\cos B - \cos A\sin B$ .....(2),

we can prove $\cos(A+B) = \cos A\cos B - \sin A\sin B$ .....(3).

130. *Given* $\sin(A+B) = \sin A \cos B + \cos A \sin B$.....(1),

to prove $\sin(A-B) = \sin A \cos B - \cos A \sin B$ .....(2).

In (1) write $A+B=D$, so that $A = D-B$.

Then we have $\sin D = \sin(D-B)\cos B + \cos(D-B)\sin B$.

But we have shown in the last article that from (1) it follows that

$$\cos(D-B) = \cos D \cos B + \sin D \sin B.$$

Substituting for $\cos(D-B)$ in above,

$\sin D = \sin(D-B)\cos B + \cos D \cos B \sin B + \sin D \sin^2 B$,

$\therefore \sin D(1-\sin^2 B) = \sin(D-B)\cos B + \cos D \cos B \sin B$,

$\therefore \sin D \cos^2 B - \cos D \cos B \sin B = \sin(D-B)\cos B$,

$\therefore \sin D \cos B - \cos D \sin B = \sin(D-B)$.

Writing A for D we have (2).

131. *Given*

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \text{.....(1)},$$

and $\sin(A-B) = \sin A \cos B - \cos A \sin B$.....(2),

to prove $\sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$ .....(5),

and $\sin A - \sin B = 2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)$ .....(6).

(1) + (2) *gives* $\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$,

(1) − (2) *gives* $\sin(A+B) - \sin(A-B) = 2 \cos A \sin B$.

Put $A+B=S$ *and* $A-B=T$,

$\therefore$ (adding) $2A = S+T$, (subtracting) $2B = S-T$,

$\therefore \sin S + \sin T = 2 \sin \frac{1}{2}(S+T) \cos \frac{1}{2}(S-T)$,

and $\sin S - \sin T = 2 \cos \frac{1}{2}(S+T) \sin \frac{1}{2}(S-T)$.

Writing A for S and B for T we have (5) and (6).

Similarly

Given $\cos(A+B) = \cos A \cos B - \sin A \sin B$.....(3),

and $\cos(A-B) = \cos A \cos B + \sin A \sin B$.....(4),

we can prove

$$\cos B + \cos A = 2 \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B) \text{.....(7)},$$

and $\cos B - \cos A = 2 \sin \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)$ .....(8).

132. The *signs* in the formulæ (1)—(8) may be connected with the fact that the *sine increases* but the *cosine decreases* as the angle increases.

133. *In* $\sin(A+B) = \sin A \cos B + \cos A \sin B$ .....(1),

and $\cos(A+B) = \cos A \cos B - \sin A \sin B$ .....(3),

and $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ .....(21).

Put $B = A$.

Thus $\sin(A+A) = \sin A \cos A + \cos A \sin A$,

i.e. $\sin 2A = 2 \sin A \cos A$ .....(9),

$\cos(A+A) = \cos A \cos A - \sin A \sin A$,

i.e. $\cos 2A = \cos^2 A - \sin^2 A$ .....(10),

$$\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A},$$

i.e. $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$ .....(17).

These results are very important.

134. *To express the sum or difference of two sines or cosines as a product.*

This is done in the general case in Arts. 124 and 125. But the following method is recommended as an exercise in the four important fundamental formulæ of Arts. 120, 121, 122, 123.

Express the two given angles as the sum and difference, respectively, of two new angles; either by inspection or by solving an equation. Then apply the formulæ of Arts. 120—123. Thus

Example 1. $\sin 7A + \sin 3A = \sin(5A+2A) + \sin(5A-2A)$
$= (\sin 5A \cos 2A + \cos 5A \sin 2A) + (\sin 5A \cos 2A - \cos 5A \sin 2A)$
$= 2 \sin 5A \cos 2A.$

Example 2. $\cos 5A - \cos 11A = \cos(8A-3A) - \cos(8A+3A)$
$= (\cos 8A \cos 3A + \sin 8A \sin 3A) - (\cos 8A \cos 3A - \sin 8A \sin 3A)$
$= 2 \sin 8A \sin 3A.$

Example 3. $\sin A - \sin B$: assuming A to be the greater.

Let $x+y=A$ *and* $x-y=B$.

Then $\sin(x+y)-\sin(x-y)=2\cos x \sin y$.

But solving for x and y, $2x=A+B$ and $2y=A-B$,

$\therefore \sin A-\sin B=2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$.

Example 4. $\cos A - \cos B$: assuming B to be the greater.

Let $x+y=B$ and $x-y=A$.

Then $\cos(x-y)-\cos(x+y)=2\sin x \sin y$.

But solving for x and y, $2x=B+A$ and $2y=B-A$.

$\therefore \cos A-\cos B=2\sin\frac{1}{2}(B+A)\sin\frac{1}{2}(B-A)$.

135. *To express the product of sines or cosines as a sum or difference.*

This is the converse problem to that of the last article. The method is simply to introduce a product containing the ratios complementary to those in the given product. Thus

Example 1.
$$2\cos 4A \sin 3A$$
$$=(\sin 4A\cos 3A+\cos 4A\sin 3A)-(\sin 4A\cos 3A-\cos 4A\sin 3A)$$
$$=\sin 7A-\sin A.$$

Example 2.
$$2\cos 7A\cos 2A$$
$$=(\cos 7A\cos 2A+\sin 7A\sin 2A)+(\cos 7A\cos 2A-\sin 7A\sin 2A)$$
$$=\cos 5A+\cos 9A.$$

EXAMPLES V.

1. Show that

$\cos(A-B)+\sin(A+B)=(\cos A+\sin A)(\cos B+\sin B)$

$\cos(A+B)+\sin(A-B)=(\cos A+\sin A)(\cos B-\sin B)$

$\cos(A-B)-\sin(A+B)=(\cos A-\sin A)(\cos B-\sin B)$

$\cos(A+B)-\sin(A-B)=(\cos A-\sin A)(\cos B+\sin B)$.

2. Find the sine and cosine of 15° from those of 60° and 45°.

3. Find the sine and cosine of 75° from those of 45° and 30°.

4. Show that

$$\tan(A+B) = \frac{\tan A \cot B + 1}{\cot B - \tan A},$$

and that

$$\cot(A-B) = \frac{\cot A + \tan B}{1 - \cot A \tan B}.$$

5. If $\sin A = \cos B = \frac{3}{5}$, find $\sin(A+B)$ and $\cos(A+B)$.

6. If

$\sin A = \frac{4}{5}\frac{5}{3}$ and $\sin B = \frac{3}{6}\frac{3}{5}$, find $\sin(A-B)$ and $\cos(A-B)$.

7. Find $\tan(A+45°)$ and $\cot(45°-A)$.

8. Show that

$$\sin A + \cos A = \sqrt{2} \cdot \sin(A+45°) = \sqrt{2}\cos(45°-A).$$

9. Show that

$$\cos A - \sin A = \sqrt{2} \cdot \sin(45°-A) = \sqrt{2}\cos(45°+A).$$

10. Find $\sin 30°$ from the identity $\cos 30° = \sin 60°$.

Prove the following identities:

11. $(\sin A + \cos A)^2 = 1 + \sin 2A$.

12. $(\cos A + \sin A)(\cos A - \sin A) = \cos 2A$.

13. $\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$.

14. $\tan^2 A = \dfrac{1-\cos 2A}{1+\cos 2A}$.

15. $\cot A = \dfrac{1+\cos 2A}{\sin 2A}$.

16. $1 + \sec 2A = 2\cos^2 A \cdot \sec 2A$.

Put the following expressions (17—28) *in the form of twice the product of two ratios:*

17. $\sin 11A + \sin 7A$.

18. $\sin 13A - \sin 9A$.

19. $\cos 3A + \cos 5A$.

20. $\cos 7A - \cos 17A$.

21. $\sin(n+1)A + \sin(n-1)A$.

22. $\cos 2A + \sin 2A$.

23. $\sin 3A - \sin 2A$.

24. $\cos 2A - \cos 3A$.

25. $\sin\theta + \cos\phi$.

26. $\cos\phi - \sin\theta$.

27. $\cos(n-3)A - \cos(n+3)A$.

28. $\sin 2A - \cos 2B$.

Put the following expressions (29—36) *in the form of the sum or difference of two ratios:*

29. $2 \sin A \cos B$.

30. $2 \sin A \sin B$.

31. $2 \sin 10° \cos 50°$.

32. $2 \sin 50° \cos 10°$.

33. $2 \cos \frac{5\theta}{2} \sin \frac{3\theta}{2}$.

34. $2 \sin \frac{7B}{2} \sin \frac{5B}{2}$.

35. $2 \sin (\alpha + \beta) \cdot \cos (\alpha - \beta)$.

36. $2 \cos 7° \sin 11°$.

Prove the following identities (37—58):

37. $\frac{\sin A - \sin B}{\cos A + \cos B} = \tan \frac{A - B}{2}$.

38. $\frac{\sin A + \sin B}{\cos A - \cos B} = \cot \frac{A - B}{2}$.

39. $\frac{\sin A + \sin B}{\cos A + \cos B} = \tan \frac{A + B}{2}$.

40. $\frac{\sin A - \sin B}{\cos A - \cos B} = \cot \frac{A + B}{2}$.

41. $\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}$.

42. $\frac{\cos A - \cos B}{\cos A + \cos B} = \frac{\tan \frac{1}{2}(A - B)}{\cot \frac{1}{2}(A + B)}$.

43. $\sin 43° + \sin 17° = \cos 13°$.

44. $\cos 81° + \cos 39° = \cos 21°$.

45. $\sin 73° - \sin 47° = \sin 13°$.

46. $\cos 11° - \cos 49° = \sin 19°$.

47. $\cos 25° - \sin 5° = \cos 35°$.

48. $\cos 5° - \sin 25° = \sin 35°$.

49. $\frac{\sin 3\theta + \sin \theta}{\cos 3\theta + \cos \theta} = \tan 2\theta$.

50. $\frac{\cos \theta - \cos 2\theta}{\sin 2\theta - \sin \theta} = \tan \frac{3\theta}{2}$.

51. $\frac{\sin \theta + \sin \phi}{\sin \theta - \sin \phi} = \tan \frac{1}{2}(\theta + \phi) \cot \frac{1}{2}(\theta - \phi)$.

52. $\dfrac{\cos\theta-\cos\phi}{\sin\theta+\sin\phi}=\dfrac{\sin\phi-\sin\theta}{\cos\theta+\cos\phi}$.

53. $\sin\dfrac{\theta}{2}\cdot\sin\dfrac{7\theta}{2}+\sin\dfrac{3\theta}{2}\cdot\sin\dfrac{11\theta}{2}=\sin 2\theta\cdot\sin 5\theta$.

54. $\tan A+\tan B=\sec A\cdot\sec B\cdot\sin(A+B)$.

55. $\cot A-\tan B=\operatorname{cosec} A\cdot\sec B\cdot\cos(A+B)$.

56. $\cos(30°-A)-\cos(30°+A)=\sin A$.

57. $\sin(45°+A)-\sin(45°-A)=\sqrt{2}\cdot\sin A$.

58. $\tan\dfrac{A+B}{2}+\tan\dfrac{A-B}{2}=\dfrac{2\sin A}{\cos A+\cos B}$.

59. If $\tan A=\frac{2}{3}$ and $\tan B=\frac{3}{7}$, find $\tan(A+B)$ and $\tan(A-B)$.

60. If $\tan A=\frac{1}{2}$ and $\tan B=\frac{1}{3}$, show that $\tan(A+B)=1$.

61. If $(1+\tan A)(1+\tan B)=2$, show that $\tan(A+B)=1$.

62. Show that if 45° is divided into two parts, θ and ϕ, then $(1+\tan\theta)(1+\tan\phi)=2$.

63. $\tan A\tan 2A\tan 3A=\tan 3A-\tan 2A-\tan A$.

64. If $\tan\theta=2m+1$ and $\tan\phi=2m-1$, then $\cot(\theta-\phi)=2m^2$.

65. $\sin(A+B)\cdot\sin(A-B)=\sin^2A-\sin^2B=\cos^2B-\cos^2A$.

66. $\cos(A+B)\cdot\cos(A-B)=\cos^2A-\sin^2B=\cos^2B-\sin^2A$.

67. $\tan(45°-A)=\dfrac{\cos 2A}{1+\sin 2A}$.

Solve:

68. $\sin 2\theta=\tan\theta$.

69. $\sin\theta=\cos 2\theta$.

70. $\cos\theta=\cos 2\theta$.

71. $\sin\theta=\sin 2\theta$.

72. $\cos 2\theta=(\sqrt{2}+1)\left(\cos\theta-\dfrac{1}{\sqrt{2}}\right)$.

73. $\cos 2\theta+3\sin\theta=2$.

74. $\tan\theta+\tan 2\theta+\sqrt{3}\tan\theta\cdot\tan 2\theta=\sqrt{3}$.

75. $\sqrt{3}(\tan\theta+\tan 2\theta)+\tan\theta\cdot\tan 2\theta=1$.

76. $\cos 2\theta-\sin\theta=\frac{1}{2}$.

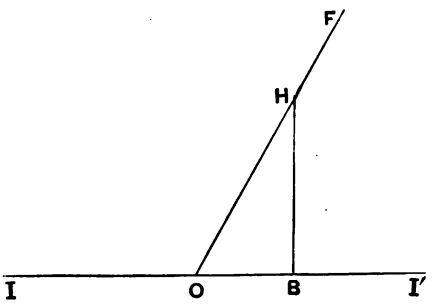
CHAPTER VI.

TRIGONOMETRICAL FORMULÆ FOR A TRIANGLE.

§ 1. The Trigonometrical ratios of Obtuse Angles.

136. We have at present defined the ratios of *acute* angles only. We now proceed to define those of *obtuse* angles.

Let IOF be any *obtuse* angle.



Produce IO to I'.

Then $I'OF$ is an *acute* angle.

In OI' take any point B: and draw BH at right angles to OI' meeting OF in H.

Then BOH is the right-angled triangle by which we determine the ratios of the *acute* angle $I'OF$, OB being base, OH being hypothenuse, and BH being perpendicular.

Now it is found convenient to use this *same* right-angled triangle BOH to define the ratios of the *obtuse* angle IOF; only that, as OB has to be drawn in the *opposite direction to the line* OI, which bounds the *obtuse* angle, we take $-OB$ (instead of OB) as *its* base. Thus

$$\sin IOF = \frac{BH}{OH}, \qquad \cos IOF = \frac{-OB}{OH},$$

$$\tan IOF = \frac{BH}{-OB}, \qquad \cot IOF = \frac{-OB}{BH},$$

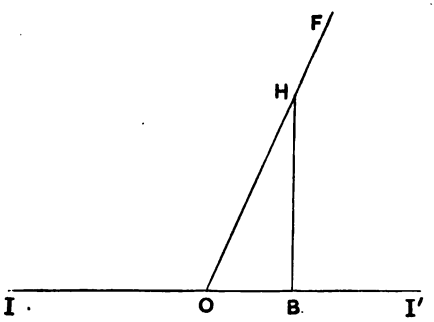
$$\sec IOF = \frac{OH}{-OB}, \qquad \operatorname{cox} IOF = \frac{OH}{BH}.$$

Thus, those ratios of an obtuse angle, *which contain the base* $-OB$ (viz. all but the sine and cosecant), are *negative*.

A fuller explanation of this adoption of the minus sign will be given later. But the student will immediately perceive its convenience, for by its means the formulæ connecting the ratios of the angles and the sides of a triangle are the *same* whether the triangle be acute-angled, right-angled or obtuse-angled.

137. *To find the ratios of the* obtuse *angles* $180° - A$ *and* $90° + A$ *in terms of those of the* acute *angle* A.

Take the figure of last article.



(1) Let $BOH = A$.

Then since $IOF + BOH = 180°$, $\therefore$ $IOF = 180° - A$,

$\sin IOF = \dfrac{BH}{OH} = \sin BOH$, i.e. $\sin (180° - A) = \sin A$,

$\cos IOF = \dfrac{-OB}{OH} = -\cos BOH$, i.e. $\cos (180° - A) = -\cos A$,

and so on.

(2) Let $BHO = A$.

Then since $IOF = OBH + BHO$, $\therefore$ $IOF = 90° + A$,

$\sin IOF = \dfrac{BH}{OH} = \cos BHO$, i.e. $\sin (90° + A) = \cos A$,

$\cos IOF = \dfrac{-OB}{OH} = -\sin BHO$, i.e. $\cos (90° + A) = -\sin A$,

and so on.

138. *To find the ratios of the* acute *angles* $180° - A$ *and* $A - 90°$ *in terms of those of the* obtuse *angle* A.

Take the same figure.

(3) Let $IOF = A$, then $BOH = 180° - A$,

$\sin BOH = \dfrac{BH}{OH} = \sin IOF$, i.e. $\sin (180° - A) = \sin A$,

$\cos BOH = \dfrac{OB}{OH} = -\cos IOF$, i.e. $\cos (180° - A) = -\cos A$,

and so on.

(4) Let $IOF = A$, then $BHO = A - 90°$,

$\sin BHO = \dfrac{OB}{OH} = -\cos IOF$, i.e. $\sin (A - 90°) = -\cos A$,

$\cos BHO = \dfrac{BH}{OH} = \sin IOF$, i.e. $\cos (A - 90°) = \sin A$,

and so on.

139. *To find the ratios of* $180°$.

In (1) of Art. 137, put $A = 0$.

Thus; $\sin 180° = \sin 0 = 0$; $\quad \cos 180° = -\cos 0 = -1$;

$\tan 180° = -\tan 0 = 0$; $\quad \cot 180° = -\cot 0 = -\infty$;

$\sec 180° = -\sec 0 = -1$; $\quad \operatorname{cox} 180° = \operatorname{cox} 0 = \infty$.

Example. To trace the changes in the ratios of obtuse *angles.*

As A increases from 90° to 180°,

$\sin A$ is $+^{ve}$; and algebraically decreases from 1 to 0.
$\cos A$ is $-^{ve}$; and algebraically decreases from 0 to -1.
$\tan A$ is $-^{ve}$; and algebraically increases from $-\infty$ to 0.
$\cot A$ is $-^{ve}$; and algebraically decreases from 0 to $-\infty$.
$\sec A$ is $-^{ve}$; and algebraically increases from $-\infty$ to -1.
$\operatorname{cox} A$ is $+^{ve}$; and algebraically increases from 1 to ∞.

140. From the proposition that *the angles of a triangle are together equal to two right angles* many important results follow.

Thus, if A, B, C are the angles of a triangle, we have

$$A + B + C = 180°,$$

$$\therefore\ B + C = 180° - A.$$

$$\therefore\ \sin(B + C) = \sin(180° - A) = \sin A, \qquad \text{[Art. 137.}$$

$$\text{and } \cos(B + C) = \cos(180° - A) = -\cos A. \qquad \text{[Art. 137.}$$

Examples VI. A.

1. Write down the ratios of 135°, 120°, 150°, $112\frac{1}{2}$°, 105°.

2. Show that $\sin 120° = 2 \sin 60° \cos 60°$ and that
$$\cos 120° = \cos^2 60° - \sin^2 60°.$$

3. Show that $\tan 150° = \cot 60° - \operatorname{cosec} 60°$ and that
$$\cos^2 135° = \cos 60°.$$

4. If A, B, C are the angles of a triangle, show that $\sin \frac{1}{2}(B + C) = \cos \frac{1}{2} A$; and that $\tan \frac{1}{2}(A + B) = \cot \frac{1}{2} C$.

5. Show that the ratios of the *half-angles* of a triangle are all positive.

6. Show that, given the sine or the cosecant of an angle of a triangle, that angle may have either of two values.

7. Show that the area of a triangle is measured by *half the product of two sides and the sine of the internal angle contained by them*, whether that angle is acute, obtuse or right.

8. Show that the ratios of any angle, whether acute or obtuse, are subject to the relations of equality established in Chapter III.

9. How must the relations of inequality of Chapter III. be modified in order to include obtuse angles?

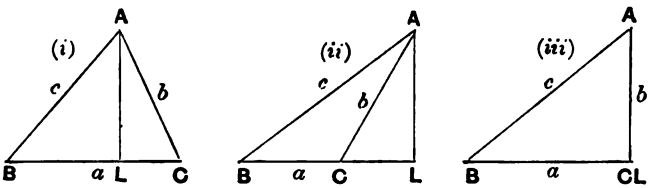
§ 2. Relations between the sides and the ratios of the Angles of a Triangle.

141. Consider any triangle ABC.

The *angles* at the points A, B, C will be called by the names A, B, C.

The *sides* BC, CA, AB, which are *opposite* to these angles, will be called a, b, c respectively.

142. *To draw the figure by which the ratios of the base angles of a triangle may be indicated.*



Draw AL perpendicular to BC or BC produced. Then, the angle ACB may be (i) acute, (ii) obtuse, or (iii) right.

In each case,

$$\sin ACB = \frac{LA}{CA}, \quad \therefore CA \sin ACB = LA.$$

But in case

(i) $\cos ACB = \dfrac{CL}{CA}$, $\quad \therefore CA \cos ACB = CL.$

(ii) $\cos ACB = \dfrac{-CL}{CA}$, $\quad \therefore CA \cos ACB = -CL.$

(iii) $\cos ACB = \dfrac{0}{CA}$, $\quad \therefore CA \cos ACB = 0.$

143. *To prove the double-cosine formula:—*

$$b \cos C + c \cos B = a.$$

In each fig. $\cos B = \dfrac{BL}{BA}$, $\quad \therefore\ BA \cos B = BL.$

In fig. (*i*) $\cos C = \dfrac{CL}{CA}$, $\quad \therefore\ CA \cos C = CL.$

In fig. (*ii*) $\cos C = \dfrac{-CL}{CA}$, $\quad \therefore\ CA \cos C = -CL.$

In fig. (*iii*) $\cos C = \dfrac{0}{CA}$, $\quad \therefore\ CA \cos C = 0.$

Again, in fig. (*i*) $BC = BL + CL,$

in fig. (*ii*) $BC = BL - CL,$

in fig. (*iii*) $BC = BL + 0.$

$\therefore$ *in each case,* $BC = CA \cos C + BA \cos B,$

i.e. $a = b \cos C + c \cos B.$

Similarly $b = c \cos A + a \cos C,$

and $c = a \cos B + b \cos A.$

144. *To prove the double-sine formula:—*

$$b \sin C = c \sin B.$$

In each fig. $\sin C = \dfrac{LA}{CA}$, $\quad \therefore\ CA \sin C = LA,$

and $\sin B = \dfrac{LA}{BA}$, $\quad \therefore\ BA \sin B = LA.$

$\therefore\ CA \sin C = BA \sin B$, i.e. $b \sin C = c \sin B.$

Cor. Dividing by bc, we have

$$\frac{\sin C}{c} = \frac{\sin B}{b}.$$

Similarly $\dfrac{\sin B}{b} = \dfrac{\sin A}{a},$

$$\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

i.e. the *sines* of the angles are proportional to the *opposite* sides.

145. By the corollary of the last article, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Put each of these fractions equal to d. Then since in each fraction the numerator is a *length* and the denominator a ratio, therefore d will represent a *length*.

Also, since $\dfrac{a}{\sin A} = d, \quad \therefore\ a = d \sin A.$

Similarly, $b = d \sin B$ *and* $c = d \sin C.$

146. *To find the sine and cosine of the sum and difference of two angles by means of the double-cosine and double-sine formulæ.*

In the equation

$$a = b \cos C + c \cos B,$$

substitute for a, b, c in terms of d. [Art. 145.

Thus $d \sin A = d \sin B \cos C + d \sin C \cos B,$

$$\therefore \sin A = \sin B \cos C + \cos B \sin C. \qquad (A.)$$

Similarly $\sin B = \sin C \cos A + \cos C \sin A.$

i.e. (substituting for $\sin A$)

$$\sin B = \sin C \cos A + \cos^2 C \sin B + \cos B \cos C \sin C,$$

$$\therefore -\sin C \cos A = \cos B \cos C \sin C - \sin B (1 - \cos^2 C),$$

$$\therefore -\cos A = \cos B \cos C - \sin B \sin C. \qquad (B.)$$

Now $\sin A = \sin (B + C)$ and $-\cos A = \cos (B + C)$, [Art. 140.

$\therefore$ (A) and (B) become

$$\sin (B + C) = \sin B \cos C + \cos B \sin C \ldots\ldots\ldots(1),$$

$$\cos (B + C) = \cos B \cos C - \sin B \sin C \ldots\ldots\ldots(2).$$

Let B' be the exterior angle at B.

Then $A = B' - C$, $\sin B = \sin B'$ and $\cos B = -\cos B'$,

$\therefore$ (A) and (B) become

$$\sin (B' - C) = \sin B' \cos C - \cos B' \sin C \ldots\ldots\ldots(3),$$

$$\cos (B' - C) = \cos B' \cos C + \sin B' \sin C \ldots\ldots\ldots(4).$$

147. It should be observed that, by the method of the last article, we have proved the formulæ for the sine and cosine of the sum and difference of two angles, *whatever geometrical value the angles considered may have*, i.e. whether they are acute or obtuse.

Hence, also, the formulæ which may be algebraically derived from these are universally true for all geometrical angles, viz. the expressions for the tangent and cotangent of the sum or difference of two angles, and the ratios of the double angles, &c.

Example. If A, B, C are the angles of a triangle, express

$$\sin A + \sin B + \sin C$$

as a product of ratios.

The given expression

$= \sin\{\frac{1}{2}(A+B)+\frac{1}{2}(A-B)\} + \sin\{\frac{1}{2}(A+B)-\frac{1}{2}(A-B)\} + \sin C$

$= 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B) + 2\sin\frac{1}{2}C\cos\frac{1}{2}C$

$= 2\cos\frac{1}{2}C\{\cos\frac{1}{2}(A-B)+\cos\frac{1}{2}(A+B)\}$,

$[\because \sin\frac{1}{2}(A+B) = \cos\frac{1}{2}C$ and $\sin\frac{1}{2}C = \cos\frac{1}{2}(A+B)]$

$= 2\cos\frac{1}{2}C \cdot 2\cos\frac{1}{2}A\cos\frac{1}{2}B$

$= 4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C.$

148. *To prove the cotan-cosec formula:—*

$$\cot B + \cot C = \frac{a}{c}\operatorname{cosec} B \text{ or } \frac{a}{b}\operatorname{cosec} C.$$

In fig. (*i*) $\cot C = \dfrac{CL}{LA}$; *in fig.* (*ii*) $\cot C = \dfrac{-CL}{LA}$.

In each fig. $\cot B = \dfrac{BL}{LA}$.

Again, in fig. (*i*), $BL + CL = BC$;

in fig. (*ii*), $BL - CL = BC$.

$\therefore$ *in each case,*

$$\cot B + \cot C = \frac{BC}{LA} = \frac{BC}{AB}\cdot\frac{AB}{LA}$$

$$= \frac{a}{c}\operatorname{cosec} B.$$

This formula may be written in other forms: thus

$$\tan B = \frac{b\sin C}{a - b\cos C} = \frac{b\tan C}{a\sec C - b}.$$

149. *To prove the cosine formula:—*

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

In fig. (*i*), (*Euc. II.* 13), $AB^2 = BC^2 + CA^2 - 2BC \,.\, CL.$

In fig. (*ii*), (*Euc. II.* 12), $AB^2 = BC^2 + CA^2 + 2BC \,.\, CL.$

In fig. (*iii*), (*Euc. I.* 47), $AB^2 = BC^2 + CA^2.$

But in fig. (*i*) $\quad CL = CA \cos C,$

in fig. (*ii*) $\quad CL = -CA \cos C,$

in fig. (*iii*) $\quad 0 = CA \cos C.$

$\therefore$ *in each case,* $\quad AB^2 = BC^2 + CA^2 - 2BC \,.\, CA \,.\, \cos C,$

i.e. $\quad c^2 = a^2 + b^2 - 2ab \,.\, \cos C.$

Similarly $\quad b^2 = c^2 + a^2 - 2ca \,.\, \cos B,$

and $\quad a^2 = b^2 + c^2 - 2bc \,.\, \cos A.$

150. *To prove the sine formula:—*

$$2ab \sin C = \surd\{(a+b+c)(a+b-c)(c+a-b)(b+c-a)\}.$$

By the last article,

$$2ab \cos C = a^2 + b^2 - c^2,$$

$\therefore$ $2ab(1 + \cos C) = (a^2 + 2ab + b^2) - c^2 = (a+b+c)(a+b-c)$ (1),

$2ab(1 - \cos C) = c^2 - (a^2 - 2ab + b^2) = (c+a-b)(c-a+b)$ (2).

$\therefore$ *multiplying the above equations* (1) and (2), we have

$4a^2b^2(1 - \cos^2 C) = (a+b+c)(a+b-c)(c+a-b)(b+c-a);$

$\therefore$ *taking the square root, since* $1 - \cos^2 C = \sin^2 C,$

$$2ab \sin C = \surd\{(a+b+c)(a+b-c)(c+a-b)(b+c-a)\}.$$

151. *To prove the half-angle formulæ.*

Since $1 + \cos C = 2\cos^2 \tfrac{1}{2}C$ *and* $1 - \cos C = 2\sin^2 \tfrac{1}{2}C,$

substituting in (1) and (2) of last article

$$4ab \cos^2 \tfrac{1}{2}C = (a+b+c)(a+b-c),$$

$$4ab \sin^2 \tfrac{1}{2}C = (c+a-b)(c-a+b).$$

Dividing, we have

$$\tan^2 \tfrac{1}{2}C = \frac{(c+a-b)(b+c-a)}{(a+b+c)(a+b-c)}.$$

152. In the formulæ of the two preceding articles, it is convenient to abbreviate by writing

$a+b+c=2s$, so that $a+b-c=2s-2c$, and so on.

Thus

$$\tfrac{1}{2}ab\sin C=\sqrt{\{s(s-a)(s-b)(s-c)\}},$$

$$\cos\frac{C}{2}=\sqrt{\left\{\frac{s(s-c)}{ab}\right\}},\quad \sin\frac{C}{2}=\sqrt{\left\{\frac{(s-a)(s-b)}{ab}\right\}},$$

$$\tan\frac{C}{2}=\sqrt{\left\{\frac{(s-a)(s-b)}{s(s-c)}\right\}}.$$

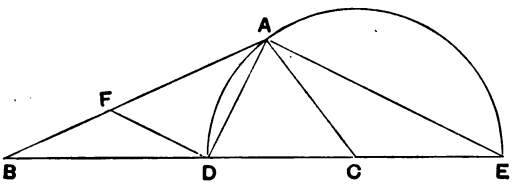
Since $\frac{1}{2}ab\sin C=$ area of triangle $=S$ (say),

$$\therefore\ S=\sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

153. *To prove the sum-and-difference formula :—*

$$\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\frac{a-b}{a+b}.$$

Assume that A is greater than B; and therefore a greater than b.



With centre C and radius CA describe a semi-circle cutting BC in D and BC produced in E.

$\therefore$ angle ADE at circumference $=\frac{1}{2}$ angle ACE at centre

$$=\tfrac{1}{2}(A+B),$$

$\therefore$ angle $DAB=ADE-ABD=\frac{1}{2}(A+B)-B=\frac{1}{2}(A-B)$.

Also angle DAE in a semi-circle is a right-angle.

Draw DF parallel to AE, or at right-angles to DA.

$$\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\frac{\tan DAF}{\tan ADE}=\frac{DF}{AD}\div\frac{AE}{AD}=\frac{DF}{AE}$$

$$=\frac{BD}{BE}\text{ by similar triangles,}$$

$$=\frac{BC-DC}{BC+CE}=\frac{a-b}{a+b}.$$

154. The student should observe that each of the identities connecting the sides and angles of a triangle (except the double-cosine formula) involves *four* out of the six elements A, B, C, a, b, c, two of the four being sides.

Thus by means of these identities we can find all the elements of a triangle when *three*, including a side, are given. [Compare Art. 29.]

In fact, since $A+B+C=180°$, we have no more specific information about any particular triangle, when 3 angles are given, than when 2 angles are given. Hence a side must always be one of the elements given.

155. The formulæ above given may be thus classified.

Class I. *Involving three angles.*

$$A+B+C=180°.$$

Class II. *Involving three sides and two angles.*

$$a=b\cos C+c\cos B.$$

Class III. *Involving two sides and the two opposite angles.*

$$a\sin B=b\sin A,$$

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\tan\frac{A+B}{2}.$$

Class IV. *Involving two sides, the included angle and another angle.*

$$c\,(\cot C+\cot B)=a\operatorname{cosec} B,$$

$$\tan C=\frac{c\sin B}{a-c\cos B}.$$

Class V. *Involving three sides and an angle.*

$$c^2=a^2+b^2-2ab\cos C,$$

$$\sqrt{\{s\,(s-a)\,(s-b)\,(s-c)\}}=\tfrac{1}{2}ab\sin C,$$

$$\sin\frac{C}{2}=\sqrt{\left\{\frac{(s-a)\,(s-b)}{ab}\right\}},$$

$$\cos\frac{C}{2}=\sqrt{\left\{\frac{s\,(s-c)}{ab}\right\}},$$

$$\tan\frac{C}{2}=\sqrt{\left\{\frac{(s-a)\,(s-b)}{s\,(s-c)}\right\}}.$$

156. The substitution of $d\sin A$, $d\sin B$, $d\sin C$ for a, b, c respectively is often useful in the working of examples on Triangular formulæ.

Example 1. Given $\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$, show that $\dfrac{b-c}{b+c}=\dfrac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}$.

Put $b=d\sin B$, $c=d\sin C$. Then

$$\frac{b-c}{b+c}=\frac{d\sin B-d\sin C}{d\sin B+d\sin C}=\frac{\sin B-\sin C}{\sin B+\sin C}$$

$$=\frac{2\sin\frac{1}{2}(B-C)\cos\frac{1}{2}(B+C)}{2\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C)}=\frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}.$$

Example 2. Show that $\tan B:\tan C=a^2+b^2-c^2:a^2-b^2+c^2$,

$$\frac{\tan B}{\tan C}=\frac{\sin B\cos C}{\sin C\cos B}=\frac{b\cos C}{c\cos B}=\frac{2ab\cos C}{2ac\cos B}=\frac{a^2+b^2-c^2}{a^2+c^2-b^2}.$$

Example 3. Show that $8\cos A\cos B\cos C$ is never greater than 1.

We have $2\cos A\cos B=\cos(A+B)+\cos(A-B)$.

Keeping $A+B$ (and, therefore, C) constant, this has its greatest value, when $\cos(A-B)=1$, i.e. when $A=B$. Hence the given expression has its greatest value, when $A=B=C=60°$, i.e. when

$$8\cos A\cos B\cos C=1.$$

Examples VI. B.

If A, B, C are the angles of a triangle, prove the following statements:—

1. $\sin(A+B+C)=\cos\frac{1}{2}(A+B+C)=0$.
2. $-\cos(A+B+C)=\sin\frac{1}{2}(A+B+C)=1$.
3. $\dfrac{\sin A-\sin B}{\sin A+\sin B}=\tan\dfrac{C}{2}\,.\tan\dfrac{A-B}{2}$.
4. $\tan A+\tan B=\sin C\,.\sec A\,.\sec B$.
5. $\tan A+\tan B+\tan C=\tan A\,.\tan B\,.\tan C$.
6. $\sin A+\sin B-\sin C=4\sin\frac{1}{2}A\,.\sin\frac{1}{2}B\,.\cos\frac{1}{2}C$.
7. $\cos A+\cos B+\cos C=4\sin\frac{1}{2}A\,.\sin\frac{1}{2}B\,.\sin\frac{1}{2}C+1$.
8. $\cos A+\cos B-\cos C=4\cos\frac{1}{2}A\,.\cos\frac{1}{2}B\,.\sin\frac{1}{2}C-1$.
9. $\sin 2A+\sin 2B+\sin 2C=4\sin A\,.\sin B\,.\sin C$.

10. $\cos 2A + \cos 2B + \cos 2C = -4 \cos A \,.\, \cos B \,.\, \cos C - 1.$

11. $\cos^2 \tfrac{1}{2}A + \cos^2 \tfrac{1}{2}B - \cos^2 \tfrac{1}{2}C = 2 \cos \tfrac{1}{2}A \,.\, \cos \tfrac{1}{2}B \,.\, \sin \tfrac{1}{2}C.$

12. $\dfrac{\cos B - \cos A}{\cos B + \cos A} = \cot \dfrac{C}{2} \,.\, \tan \dfrac{A-B}{2}.$

13. $\tan \tfrac{1}{2}B \,.\, \tan \tfrac{1}{2}C + \tan \tfrac{1}{2}C \,.\, \tan \tfrac{1}{2}A + \tan \tfrac{1}{2}A \,.\, \tan \tfrac{1}{2}B = 1.$

14. $\sin^2 A = \cos^2 B + \cos^2 C + 2 \cos A \,.\, \cos B \,.\, \cos C.$

15. $\cos^2 A + \cos^2 B + 2 \cos A \cos B \cos C$
$= \sin^2 A + \sin^2 B - 2 \sin A \sin B \cos C.$

16. $8 \sin \tfrac{1}{2}A \,.\, \sin \tfrac{1}{2}B \,.\, \sin \tfrac{1}{2}C$ is never greater than 1.

In any triangle ABC prove the following statements:—

17. $\tan A = \dfrac{a \sin C}{b - a \cos C}.$

18. $b (\tan B + \tan C) = a \tan B \sec C.$

19. $a \cos A + b \cos B + c \cos C = 2a \sin B \sin C.$

20. $(a+b) \cos C + (b+c) \cos A + (c+a) \cos B = a + b + c.$

21. $a (b \cos C - c \cos B) = b^2 - c^2.$

22. $a \sin^2 B = b (\cos A \cos B + \cos C).$

23. $a^2 + b^2 + c^2 = 2ab \cos C + 2bc \cos A + 2ca \cos B.$

24. $ab \sin^2 C = c (a \cos B \cos C + b \cos C \cos A + c \cos A \cos B).$

25. $4S = a (2b \sin B \cos A + a \sin 2B).$

26. $2b (1 - \sin B \cos A \operatorname{cosec} C) = a \sin 2B \operatorname{cosec} C.$

27. $2 \cos C (a \sin A - b \sin B) = c (\sin 2B - \sin 2A).$

28. $bc \cos A + ca \cos B + 2ab \cos C = a^2 + b^2.$

29. $b \sin 2A = 2a (\sin C - \cos B \sin A).$

30. $S (a + b + c) = abc (\sin B \cos^2 \tfrac{1}{2}A + \sin A \cos^2 \tfrac{1}{2}B).$

31. $(a - b) \cos \tfrac{1}{2}C = c \sin \tfrac{1}{2}(A - B).$

32. $a^2 \cos B + b^2 \cos C + c^2 \cos A$
$= 2ab \sin^2 \tfrac{1}{2}A + 2bc \sin^2 \tfrac{1}{2}B + 2ca \sin^2 \tfrac{1}{2}C.$

33. $(b^2 - c^2) \cos A + (c^2 - a^2) \cos B + (a^2 - b^2) \cos C$
$= a \cos A (b - c) + b \cos B (c - a) + c \cos C (a - b).$

34. $s (2ab + 2bc + 2ca - a^2 - b^2 - c^2)$
$= 2abc (\cos^2 \tfrac{1}{2}A + \cos^2 \tfrac{1}{2}B + \cos^2 \tfrac{1}{2}C).$

35. $\dfrac{\tan A}{\tan B} = \dfrac{a^2 + c^2 - b^2}{b^2 + c^2 - a^2}$.

36. $\tan A\,(b^2 + c^2 - a^2) = 4S$.

37. $a \sec A - b \sec B = \sec C\,(b \sec A - a \sec B)$.

38. $Ss = abc\,(\sin A + \sin B + \sin C)$.

39. $\dfrac{(s-a)\sin^2 \frac{1}{2} A}{a} = \dfrac{(s-b)\sin^2 \frac{1}{2} B}{b} = \dfrac{(s-c)\sin^2 \frac{1}{2} C}{c} = \dfrac{S^2}{sabc}$.

40. $a \sin B + b \sin C + c \sin A$

$$= \cos \frac{A}{2}\left(c \cos \frac{B+C}{2} + a \cos \frac{B-C}{2}\right) + \cos \frac{B}{2}\left(c \cos \frac{A+C}{2} + b \cos \frac{A-C}{2}\right).$$

If C is a right angle, prove the following statements:—(41—50)

41. $\tan B = \cot A$.

42. $\tan A + \tan B = \sec A \sec B$.

43. $c + a : b = b : c - a$.

44. $\tan 2A + \tan 2B = 0$.

45. $\cot \dfrac{A}{2} = \dfrac{b+c}{a}$.

46. $\cos^2 \dfrac{A}{2} = \dfrac{b+c}{2c}$.

47. $\sec 2B = \dfrac{a^2 + b^2}{a^2 - b^2}$.

48. $\operatorname{cosec} 2B = \dfrac{a}{2b} + \dfrac{b}{2a}$.

49. $\tan 2B = \dfrac{2 \sin A \sin B}{\sin^2 A - \sin^2 B}$.

50. $4S^2 = abc^2 \cos A \cos B$.

51. If any one of the above equations (41—50) holds, examine in each case whether C is a right angle.

52. If, in any triangle ABC, d, e, f are the distances of the angles from the middle points of the opposite sides,

$$4\,(d^2 + e^2 + f^2) = 3\,(a^2 + b^2 + c^2).$$

53. If l, m, n are the perpendiculars from A, B, C on the opposite sides,

$$2\,(l \cos A + m \cos B + n \cos C) = a \sin A + b \sin B + c \sin C.$$

54. If BC be bisected in D and produced to E, $\cot A$ is the Arithmetic Mean between $\cot DAC$ and $\cot ACE$.

55. If D be the middle point of BC, H the point where the bisector of A cuts BC, L the foot of the perpendicular from A on BC; then

$$4DH \,.\, DL = (b \sim c)^2.$$

56. If the base of a triangle be trisected, and k_1, k_2, k_3 be the cotangents of the angles which the parts subtend at the vertex,

$$(k_1 + k_2)(k_2 + k_3) = 4(1 + k_2^2).$$

57. If a, b, c are in A. P., then also $\cos A$, vers B, $\cos C$ are in A. P.

58. If a^2, b^2, c^2 are in A. P., then also $\cot A$, $\cot B$, $\cot C$ are in A. P.

59. If $\frac{b+c}{a}$, $\frac{c+a}{b}$, $\frac{a+b}{c}$ are in A. P., then also

$\cot A \cot \frac{1}{2}A$, $\cot B \cot \frac{1}{2}B$, $\cot C \cot \frac{1}{2}C$ are in A. P.

60. If $a^2 + bc$, $b^2 + ca$, $c^2 + ab$ are in A. P., then also $\tan \frac{1}{2}A$, $\tan \frac{1}{2}B$, $\tan \frac{1}{2}C$ are in A. P.

61. If $a^4 + b^2c^2$, $b^4 + c^2a^2$, $c^4 + a^2b^2$ are in A. P., then also $\tan A$, $\tan B$, $\tan C$ are in A. P.

62. If AH be the bisector of A,

$$AH^2 . (b + c)^2 = 4bcs(s - a).$$

63. If x, y, z are the perpendiculars on the sides from any point within a triangle,

$$ax + by + cz = 2S.$$

64. In the last proposition show that $x^2 + y^2 + z^2$ is a minimum, when $x/a = y/b = z/c = 2S/(a^2 + b^2 + c^2)$.

[In Examples (65—69), the letters a, b, c, A, B, C are not *given* as the elements of a triangle.]

65. Given that $a = b\cos C + c\cos B$, $b = c\cos A + a\cos C$, $c = a\cos B + b\cos A$, solve for $\cos A$, $\cos B$, $\cos C$.

66. From the same equations, show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{2\sqrt{\{s(s-a)(s-b)(s-c)\}}}{abc},$$

where $2s = a + b + c$.

67. Given that

$$A + B + C = 180°, \quad b = c\cos A + a\cos C, \quad c = a\cos B + b\cos A,$$

show that $a = b\cos C + c\cos B$.

68. Given that

$$b \sin A = a \sin B,\ c = a \cos B + b \cos A,\ 2ab \cos C = a^2 + b^2 - c^2,$$

show that one value of $A + B + C$ is 180°, and that

$$b \sin C = c \sin B,\ a = b \cos C + c \cos B.$$

69. Given that $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ and $c = a \cos B + b \cos A$, find $\sin A$, $\sin B$, $\sin C$ in terms of a, b, c.

70. If D be the middle point of BC, and if AD and AD' make the same angle with the bisector of BAC, then $BD' : D'C = c^2 : b^2$ and $AD' : AD = 2bc : b^2 + c^2$.

71. If L be the foot of the perpendicular from A on BC, and if AL and AL' make the same angle with the bisector of BAC, then, (A, B, C being in descending order of magnitude,)

$$\frac{AL}{AL'} + \frac{BM}{BM'} + \frac{CN}{CN'} = 4 \cos \frac{A-B}{2} \cos \frac{B-C}{2} \cos \frac{A-C}{2} - 1.$$

72. If D and E be points in BC such that AD and AE make equal angles with the bisector of BAC, then

$$BD \,.\, BE : CD \,.\, CE = c^2 : b^2.$$

73. If $b + c : c + a : a + b = 4 : 5 : 6$, then
$\sin A : \sin B : \sin C = 7 : 5 : 3$, $\cos A : \cos B : \cos C = -7 : 11 : 13$ and $A = 120°$.

CHAPTER VII.

SOLUTION OF TRIANGLES [WITHOUT LOGARITHMS].

157. We have shown in Art. 29 that, when the values of **three** elements of a triangle including **one length** are given, the other elements of the triangle have a determinate value. The process of finding the unknown elements of a triangle, when a sufficient number of elements are given, is called the *Solution of Triangles.*

158. The solution is accomplished by means of the formulæ of the last chapter. Thus, from an equation involving *four* elements, the fourth element may be found when *three* of them are known.

159. We shall have to investigate in each case (1) whether or not the solution found represents *any possible* geometrical triangle, and (2) whether or not the solution gives *more than one possible* geometrical triangle.

160. The condition, algebraically considered, for a possible solution is that any quantity whose square root has to be taken must be positive.

The conditions, geometrically considered, for a possible solution are that any length, any angle, and the ratio of any angle (except the cos, tan, sec, or cot of such angles as may be obtuse) must be represented in the solution by a *positive* value; and that the sine or cosine of any angle must be represented in the solution by a value *numerically not greater than* 1.

Of course it will be assumed that the elements which are *given* are themselves possible.

161. There will be more than one solution, algebraically considered, wherever a square root has to be taken; for before this square root we may place a positive or negative sign.

There will be more than one solution, geometrically considered, wherever an angle has to be found from its sine or cosecant; for such an angle may have either one of two values, supplementary to one another.

Of course one or other of such alternative solutions must be rejected if it involves an impossible value for some other element.

162. The chief cases to be considered in the solution of triangles are those which correspond to the cases of exact equality of two triangles given in Art. 29. These we will now consider.

Case I.

Given two angles and a side: as B, C, a.

163. First, to find A.

Since $A+B+C=180°$, $\therefore\ A=180°-B-C$.

This determines for A a real, i.e. positive, value provided $B+C$ is less than 180°. (Compare Euc. I. 17.)

Next, to find b or c.

Since $$\frac{b}{\sin B}=\frac{a}{\sin A},\quad \therefore\ b=\frac{a\sin B}{\sin A}.$$

Since $$\frac{c}{\sin C}=\frac{a}{\sin A},\quad \therefore\ c=\frac{a\sin C}{\sin A}.$$

Since all the values involved are positive, b and c have each one real, i.e. positive, value.

164. Or, we may find b or c without first finding A. For

since $$\cot B+\cot C=\frac{a}{b}\operatorname{cosec} C,\quad \therefore\ b=\frac{a\operatorname{cosec} C}{\cot B+\cot C}.$$

If b, instead of a, had been given, the equations for finding a and c would be

$$a=b\sin C\,(\cot B+\cot C),\ \textit{and}\ c=\frac{b\sin C}{\sin B}.$$

Case II.

Given three sides: a, b, c.

165. Since $2bc \cos A = b^2 + c^2 - a^2$,

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

This determines for A a real value, provided the above fraction is *numerically not greater than* 1.

(1) Let the fraction be *positive*, i.e. $b^2 + c^2 > a^2$.

Then we must have

$$2bc > b^2 + c^2 - a^2, \text{ i.e. } a^2 > (b-c)^2,$$

i.e. (assuming b not $< c$) $a > b - c$, i.e. $a + c > b$.....(2).

Moreover, since b not $< c$, $\therefore$ *a fortiori*, $a + b > c$.....(3);

and, since $b^2 + c^2 > a^2$, $\therefore$ *a fortiori*, $b + c > a$.....(1).

(2) Let the fraction be *negative*, i.e. $a^2 > b^2 + c^2$.

Then we must have

$$2bc > a^2 - b^2 - c^2, \text{ i.e. } (b+c)^2 > a^2, \text{ i.e. } b + c > a.....(1).$$

Moreover,

$\because a^2 - c^2 > b^2$, but $a - c < b$, $\therefore a + c > b$.....(2),

$\because a^2 - b^2 > c^2$, but $a - b < c$, $\therefore a + b > c$.....(3).

Hence the condition that any one angle A may be real is that, of the sides given, *any two must be together greater than the third.* (Compare Euc. I. 20.)

166. Or, we may solve by means of the equation

$$\tan \frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}}.$$

This determines for $\tan \frac{1}{2} A$ a real value, provided that the fraction whose root has to be taken is *positive:* i.e. provided that *two* or *none* of the factors $s - a$, $s - b$, $s - c$ are negative.

But *two* of the factors, such as $s - a$ and $s - b$, cannot be negative, for then $2s - a - b$, i.e. c would be negative.

Hence *none* must be negative: i.e.

$$2s > 2a \text{ or } b+c > a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$
$$2s > 2b \text{ or } c+a > b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$
$$2s > 2c \text{ or } a+b > c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Moreover, since A must be $< 180°$, $\frac{1}{2}A$ must be $< 90°$, therefore the *positive* value of the root must be taken for $\tan \frac{1}{2}A$.

Hence if the conditions (1), (2), (3) are *not* fulfilled there is *no* possible solution: if they *are* fulfilled, there is *one* possible solution.

Case III.

Given two sides and the included angle: as b, c, A.

167. First, to find a.

Since $a^2 = b^2 + c^2 - 2bc \cos A$, $\therefore\ a = \sqrt{(b^2 + c^2 - 2bc \cos A)}$.

Taking the positive value of this root, we have one real value for a, provided that

$b^2 + c^2 - 2bc \cos A$, i.e. $(b-c)^2 + 2bc\,(1 - \cos A)$ is positive.

Now since b, c, and A are assumed to be real, $(b-c)^2$ is positive and $1 - \cos A$ is positive. Hence there is *always* one real value for a.

Next, to find B and C, we have

$$\cot B = \frac{c}{b} \operatorname{cosec} A - \cot A \text{ and } \cot C = \frac{b}{c} \operatorname{cosec} A - \cot A,$$

which give *one* real value for B and C.

168. Or, to find B and C, (assuming $b > c$) we may use

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \tan \frac{B+C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}.$$

This gives us $\frac{1}{2}(B-C)$; and since $\frac{1}{2}(B+C) = 90° - \frac{1}{2}A$, by addition we have B, and by subtraction we have C.

Lastly, $\qquad a = \dfrac{b \sin A}{\sin B}$ gives us a.

CASE IV.

Given two sides and a non-included angle: as b, c, B.

169. Here C has to be determined from the equation

$$\sin C = \frac{c \sin B}{b}.$$

(I) *Let* $b < c \sin B$, so that $\frac{c \sin B}{b} > 1$.

Then there is no value of C, such that sin C has the required value.

$\therefore$ there is *no* solution.

(II) *Let* $b = c \sin B$, so that $\frac{c \sin B}{b} = 1$.

Then $\sin C = 1, \quad \therefore C = 90°,$

and $A = 180° - B - C = 90° - B.$

(i) *Let* $B = or > 90°$.

Then A is zero or negative.

$\therefore$ there is *no* solution.

(ii) Let $B < 90°$.

Then A is positive,

and $a = c \cos B$, which is positive.

$\therefore$ there is *one* solution.

(III) *Let* $b > c \sin B$, so that $\frac{c \sin B}{b} < 1$.

Then there are *two* values of C, supplementary to one another, whose sines have the required value.

$\therefore C = C_1$ (acute) *or* $180° - C_1$ (obtuse) say,

and $A = 180° - B - C_1$ *or* $C_1 - B.$

(i) *Let* $B = or > 90°$.

Then the second value of A is negative.

Also we have $\frac{\sin (180° - B)}{\sin C_1} = \frac{b}{c}$.

(1) *Let* $b = \text{or} < c$.

Then $180° - B = \text{or} < C_1$.

$\therefore$ the first value of A is zero or negative.

$\therefore$ there is *no* solution.

(2) *Let* $b > c$.

Then $180° - B > C_1$.

$\therefore$ the first value of A is positive,

and $a = \dfrac{b \sin A}{\sin B}$, which is positive.

$\therefore$ there is *one* solution.

(ii) Let $B < 90°$.

Then the first value of A is positive.

This gives $a = \dfrac{b \sin A}{\sin B}$, which is positive.

Also we have $\dfrac{\sin B}{\sin C_1} = \dfrac{b}{c}$.

(1) *Let* $b = \text{or} > c$.

Then $B = \text{or} > C_1$.

$\therefore$ the second value of A is zero or negative.

$\therefore$ there is only *one* solution.

(2) Let $b < c$.

Then $B < C_1$.

$\therefore$ the second value of A is positive.

$\therefore$ there are *two* solutions.

170. Summing up the above results:

There is *no* solution, if

(1) *either* $b < c \sin B$, [I.]

(2) *or* b not $> c$ and B not acute. [I., II. i., III. i. 1.]

There is *one* solution, if.

(3) *either* $b > c$, [III. i. 2, III. ii. 1.]

(4) *or* $b = c$ or $c \sin B$ and B acute. [III. ii. 1, II. ii.]

There are *two* solutions, if

(5) $b < c$ *but* $> c \sin B$ and B acute. [III. ii. 2.]

171. The really **Ambiguous Case** is, therefore, that in which

Two sides and an angle opposite one of them are given:—the given angle being acute, and the side opposite it being smaller than the other given side, but greater than the product of that side into the sine of the given angle.

172. We might also find a without finding A or C.

Thus, since $b^2 = c^2 + a^2 - 2ac \cos B$,

solving for a, we have

$$a^2 - 2ac \cos B + c^2 \cos^2 B = b^2 - c^2 + c^2 \cos^2 B$$
$$= b^2 - c^2 \sin^2 B,$$
$$\therefore\ a = c \cos B \pm \sqrt{(b^2 - c^2 \sin^2 B)}.$$

If then $b < c \sin B$, the values of a are algebraically impossible.

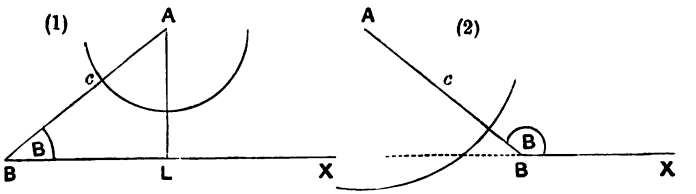
If $b = c \sin B$, there is *one* value for a.

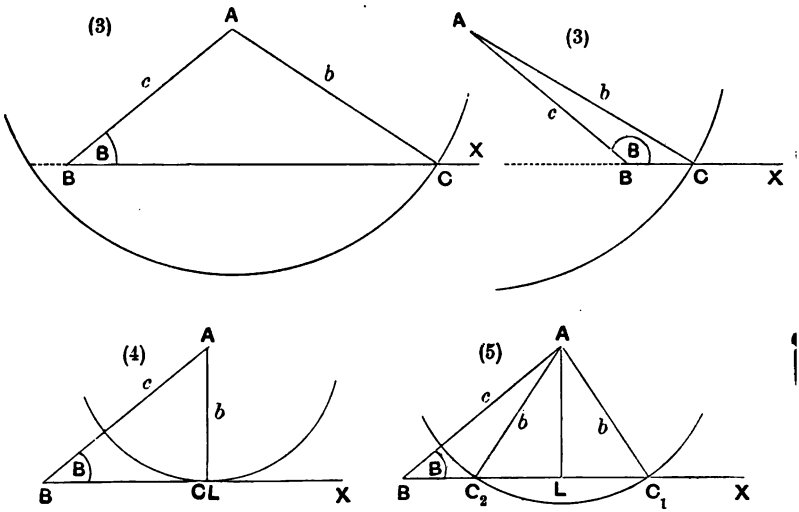
If $b > c \sin B$, there are *two* values of a, which are *both* positive, if $c^2 \cos^2 B > b^2 - c^2 \sin^2 B$, i.e. if $c^2 > b^2$.

173. The student should observe the *geometrical* illustration of the five cases considered in Art. 170. Thus

Let AB be drawn of the given length c: make the angle ABX of the given value B. Then with centre A and radius equal to the given length b, describe a circle. The point or points, if any, in which this circle cuts BX will give the third angular point C.

Drawing AL perpendicular to BX, $AL = c \sin B$.





In fig. (1), where $b < AL$, the circle does not cut BX at all.

In fig. (2), where B is obtuse, and $b < c$, the circle does not cut BX at all on the side on which the angle B is formed *.

In figs. (3), where $b > c$, the circle cuts BX in only *one* point on the side on which the angle B is formed *.

In fig. (4), where $b = AL$ and B is acute, the circle *touches* BX at one point L.

In fig. (5), where b lies in magnitude between AB and AL, and B is acute, the circle cuts BX in *two* points on the side on which the angle B is formed.

Other cases for solution.

174. The four cases above considered are the most important. But a triangle may be solved from other data.

* If $b = c$, the circle cuts BX at B, and the triangle collapses into a straight line; giving no solution when B is not acute, and one solution when B is acute.

It is always necessary that the value of *one* length at least should be given.

A few examples are worked out below.

Example 1. Given $C=120°$, $a=3$, $b=5$, find c.

We have $$c^2=a^2+b^2-2ab\cos C.$$

Now $$\cos C=\cos 120°=-\cos 60°=-\tfrac{1}{2},$$
$$\therefore\ c^2=a^2+b^2+ab=3^2+5^2+3\,.\,5=49.$$
$$\therefore\ c=7.$$

Example 2. Given $A=30°$, $B=45°$, $a=\sqrt{2}$, find b and c.

$$b=\frac{a\sin B}{\sin A}=\frac{2a}{\sqrt{2}}=a\sqrt{2}=2,$$

$$c=a\cos B+b\cos A=\frac{\sqrt{2}}{\sqrt{2}}+\frac{2\sqrt{3}}{2}=1+\sqrt{3}.$$

Example 3. Given $B=30°$, $b=3\sqrt{2}$, $c=6$, solve the triangle.

$$\sin C=\frac{c\sin B}{b}=\frac{6}{3\sqrt{2}\times 2}=\frac{1}{\sqrt{2}},\quad\therefore\ C=45°\text{ or }135°.$$
$$\therefore\ A=180°-B-C=105°\text{ or }15°,$$

and $$a=c\cos B+b\cos C=\frac{6\sqrt{3}}{2}+\frac{3\sqrt{2}}{\sqrt{2}}\text{ or }\frac{6\sqrt{3}}{2}-\frac{3\sqrt{2}}{\sqrt{2}}$$
$$=3(\sqrt{3}+1)\text{ or }3(\sqrt{3}-1).$$

Example 4. If a_1 and a_2 are the two values of a, when B, b, c are given, show that $a_1+a_2=2c\cos B$.

We have $$a_1=c\cos B+b\cos C_1,$$
and $$a_2=c\cos B+b\cos C_2,$$
where C_1, C_2 are supplementary, so that $\cos C_1=-\cos C_2$,
$$\therefore\ \text{adding } a_1+a_2=2c\cos B.$$

EXAMPLES VII.

1. Given $A=60°$, $b=7$ ft., $c=5$ yds., find a.
2. Given $a=\sqrt{13}$, $b=3$, $c=4$, find A.
3. Given $\sin C=\cdot 6$, $b=5$, $a=7$, find B.
4. Given $C=45°$, $\cos B=\cdot 28$, $c=5$, find a, b, and $\sin A$.
5. Given $A=30°$, $b=4$ ft., $a=1$ yd., find c.
6. Given $a=2$, $b=\sqrt{6}$, $c=1+\sqrt{3}$, find A, B, C.

7. Given $a = 2$, $b = \sqrt{2}$, $c = \sqrt{3} - 1$, find A, B, C.

8. Given $B = 120°$, $b = 14$, $c = 10$, find $\cos C$ and a.

9. Given $a : b : c = 44 : 117 : 125$, and that
$$\sin 20° 36' 35'' = \cdot 352,$$ find A, B, C.

10. Given $C = 18°$, $a - c = 2$, $ac = 4$, find A and B.

11. Given $\sin C + \cos C = \frac{a}{b}$, find B.

12. Given $\cos C = \frac{1}{14}$, $\cos A = \frac{17}{22}$, find $a : b : c$.

13. Given $a = 10$ ft., $b = 15$ ft., $C = 30°$, find $\cot A$ and the area of ABC.

14. Given $a = 18$, $b = 16$, $c = 14$, find the distance of B from the middle point of AC.

15. Given $S = 6$ sq. miles, $a = 3$ m., $b = 5$ m., find c.

16. Given $A = 30°$, and $b : c = 1 : \sqrt{3}$, solve the triangle.

17. Given $(a + b + c)(b + c - a) = 3bc$, find A.

18. If $a \cos A = b \cos B$, show that the triangle is either isosceles or right-angled.

19. Given $a = 193$ ft., $b = 194$ ft., $c = 195$ ft., find $\sin A$, $\sin B$, $\sin C$.

20. If $b \cos A = a \cos B$, show that the triangle is isosceles.

21. Given $A = 75°$, $C = 45°$, and the perpendicular from A on BC is 3 ft., solve the triangle.

22. Given $\tan C \cot B = 5$, $\tan B \tan C = 9$, and the distance of A from the middle point of BC is 7 in., find BC.

23. If the base-angles of a triangle are $22\frac{1}{2}°$ and $112\frac{1}{2}°$, show that its base is twice its height.

24. The sides of a triangle are in A. P., and the difference between the cotangents of the halves of the greatest and least angles is 4. Show that the cosine of the remaining angle is $\frac{3}{4}$.

CHAPTER VIII.

HEIGHTS AND DISTANCES.

175. THE formulæ of Chapter VI. connecting the sides and angles of a triangle are of practical use in enabling us to calculate lengths and angles, which cannot be directly measured.

Many objects in space are absolutely or practically inaccessible, though visible to the eye. In order to determine the distances of such objects from any others we require to measure directly

(1) The *distance* between some pair of *accessible* points.

(2) The *angle* between the lines drawn from the eye to some pair of *visible* points.

By means of suitable instruments an angle can be measured with greater accuracy than a length. Hence it is usual to make our calculations depend on the measurement of a *single length* and of as many angles as are necessary.

176. The angle made *with the horizontal plane* by the line joining the eye to an object is called its **angle of elevation** [or, briefly, its *elevation*] when it is *above* the observer; its **angle of depression** [or, briefly, its *depression*] when it is below the observer.

177. *To express the relation between the* (*vertical*) height *and the* (*horizontal*) distance *of an object.*

Let P be any object, A the position of the observer, PN the vertical line through P, and AN the horizontal line through A drawn to meet PN. (See first fig. of Art. 179.)

The angle NAP is the angle of elevation of P, and the angle ANP is a right angle: thus

$$\frac{AN}{NP} = \cot NAP, \quad \therefore \text{ distance} = \text{height} \times (\text{cot of elevation}),$$

$$\frac{NP}{AN} = \tan NAP, \quad \therefore \text{ height} = \text{distance} \times (\text{tan of elevation}).$$

These results are very important.

Example 1. The angle of elevation of a tower 130 ft. high is observed to be 60°. Find the distance of the tower from the point of observation.

Here the angle $NAP = 60°$, and $NP = 130$ ft.,

$$\therefore AN = NP \cdot \cot NAP = 130 \text{ ft.} \times \cot 60°$$
$$= 130 \text{ ft.} \times \frac{1}{\sqrt{3}} = \frac{130 \times \sqrt{3}}{3} \text{ ft.} = \frac{130 \times 1{\cdot}7320}{3} \text{ ft.}$$
$$= 57{\cdot}73 \text{ ft.} + 17{\cdot}320 \text{ ft.}$$
$$= 75 \text{ ft. nearly.}$$

Example 2. A man, standing on the top of a tower and looking in a direction perpendicular to the length of a river, which is 140 ft. wide, observes that the angles of depression of the two banks of the river are 45° and 30°. Find the height of the tower.

Let A be the further, B the nearer bank of the river; P, the point of observation; and PN the vertical through P. (See fig. of Art. 179.)

Then angle $PBN = 45°$, and angle $PAN = 30°$; and $AB = 140$ ft.

Now $\because PBN = 45°$ and $PNB = 90°$, $\therefore BN = PN$.

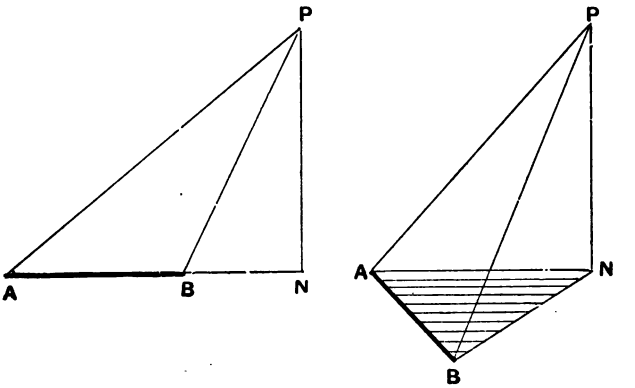
Let PN or $BN = x$. Then $AN = AB + BN = 140 \text{ ft.} + x$,

and
$$\frac{AN}{PN} = \cot 30°, \text{ i.e. } \frac{x + 140 \text{ ft.}}{x} = \sqrt{3},$$

$$\therefore x = \frac{140 \text{ ft.}}{\sqrt{3} - 1} = \frac{140(\sqrt{3} + 1)}{3 - 1} \text{ ft.} = 70 \times 2{\cdot}7320 \text{ ft.}$$
$$= 191{\cdot}24 \text{ ft.} = 191\tfrac{1}{4} \text{ ft. nearly.}$$

178. The values of the trigonometrical ratios of any angles are to be found in published tables. If any ratio cannot be found by the elementary geometrical methods of Chapter IV. reference must be made to these tables.

179. *To find the height and distance of an inaccessible object.*



Let P be the top of an object, PN the vertical through P.

Let A, B be two points of observation, the distance between which is measured.

(1) Suppose ABN to be in a *straight line*, so that A, B, N, P are in the same vertical plane.

At A and B measure the angles of elevation NAP, NBP.

Then the angle PBA is the supplement of NBP.

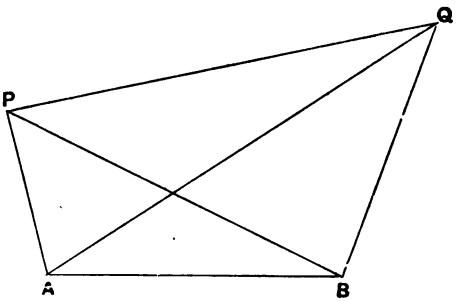
(2) Suppose ABN not to be a straight line.

At A measure the angles NAP, BAP; and at B measure the angle PBA.

Then, both in (1) and (2), in the triangle PAB, the side AB and the angles PAB, PBA are known, $\therefore$ *AP can be calculated.*

$\therefore$ in the right-angled triangle PAN, AP and $\angle NAP$ are known, $\therefore$ *AN and PN can be calculated.*

180. *To find the distance between two visible objects.*



Let P and Q be the two objects; A and B two points of observation.

Measure AB.

At A, measure the angles PAB, QAB.

At B, measure the angles PBA, QBA.

Also if the triangles PAB, QAB are not in the same plane, measure the angle PAQ.

Then, in triangle PAB; AB, $\angle PAB$, and $\angle PBA$ are known; $\therefore$ *AP can be calculated.*

In triangle QAB; AB, $\angle QBA$, and $\angle QAB$ are known; $\therefore$ *AQ can be calculated.*

Lastly, in triangle PAQ; AP, AQ and $\angle PAQ$ are known; $\therefore$ *PQ can be calculated.*

181. In working problems, which involve points not all in one plane, it is often useful to employ a figure in which, for points not in the observer's horizontal plane, are substituted the feet of the perpendiculars from them on that plane.

Thus, if P is any point not in the horizontal plane containing two points of observation A and B, draw PN perpendicular upon that plane.

Then (by Art. 177) we have

$AN = PN \times$ (cot of elevation of P at A),

$BN = PN \times$ (cot of elevation of P at B).

Example. At A and B, the angles of elevation of an object P are observed to be α and β respectively. The distance between A and B is c: and the angle between AB and the line joining A to the foot of P is θ. Find the height of P.

Let N be the foot of P, and $PN=x$.

Then in the triangle ABN we have

$$AB=c,\ AN=x\cot\alpha,\ BN=x\cot\beta\ ;\ \angle BAN=\theta.$$

Thus $$BN^2=AB^2+AN^2-2AB\,.\,AN\,.\cos\theta,$$

i.e. $$x^2\cot^2\beta=c^2+x^2\cot^2\alpha-2cx\cot\alpha\cos\theta,$$

$$\therefore\ x^2(\cot^2\beta-\cot^2\alpha)+2cx\cot\alpha\cos\theta=c^2,$$

a quadratic equation giving x.

EXAMPLES VIII.

1. At a point 866 ft. from the base of a column, the angle of elevation of its summit is observed to be 30°. Find the height of the column.

2. From the top of a hill the angle of depression of an object on the ground is observed to be 60°. The hill being 1732 ft. high, find the distance of the object from the point where the vertical line through the top of the hill would cut the ground.

3. A man 5 ft. 10 in. high observes that his shadow from a lamp 3 yds. 4 in. high is 1 yd. 9 in. What is his distance from the lamp?

4. A tower 50 ft. high stands on a mound; from a point on the ground the angles of elevation of the top and bottom of the tower are observed to be 75° and 45° respectively; find the height of the mound.

5. The angle of elevation of a balloon from a station due north of it is 45°, and from a station at a mile due east of the former station it is 30°. Find the height of the balloon and its distance in a straight line from the second station.

6. A man, walking along a straight road at the rate of $3\frac{1}{2}$ miles an hour, observes that a house, whose direction $\frac{1}{4}$ hr. ago made an angle of 45° with the road, is now directed at an angle

of 75°. Find how long he will take to reach the nearest point on the road to the house and how distant the house will then be.

7. At two points, 400 ft. apart, on the bank of a straight river, the direction of an object on the other bank is observed to make with the first bank angles equal to 72° and 36° respectively. Find the breadth of the river, and the distance of the object from the first point of observation.

8. At two points A and B, at a distance c apart, the angles of elevation of an object P are observed to be α and β respectively. The straight line AB is horizontal and the plane ABP is vertical. Show that the height of P above the horizontal plane through AB is

$$\frac{c}{\cot\alpha \sim \cot\beta}.$$

9. At two points A and B, at a distance c apart, the angles of elevation of an object P are observed to be α and β respectively. The plane ABP not being vertical, it is observed that the difference of the angles BAP and ABP is 90°. Show that the height of P is

$$\frac{c\sqrt{(\operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta)}}{\operatorname{cosec}^2\alpha \sim \operatorname{cosec}^2\beta}.$$

10. An object $2b$ feet high, placed on the top of a tower, subtends an angle α at a place whose horizontal distance from the foot of the tower is b feet; show that the height of the tower is

$$b\{\sqrt{(2\cot\alpha)} - 1\}.$$

11. The angles of elevation of a tower from three points A, B, C in a straight line are observed to be α, β, γ respectively. If $BC = a$, $AC = b$, $AB = c$, show that the height of the tower is

$$\sqrt{\left(\frac{abc}{a\cot^2\alpha - b\cot^2\beta + c\cot^2\gamma}\right)}.$$

12. A house of three equal storeys is observed from three points A, B, C in a straight line. It is found that (when looking at a fixed vertical line on the house) the angle of elevation of the top of the lowest storey at A is equal to that of the top of the

middle storey at B and to that of the top of the highest storey at C. If $AB = a$, $BC = b$, show that the distance of A from the foot of the vertical line is

$$\sqrt{\left\{\frac{ab\,(a+b)}{5a-3b}\right\}}.$$

13. In the last question find the cosines of the angles subtended by AB, BC, and AC respectively at the foot of the vertical line; and show that if $AB = 2\,.\,BC$ and if α, β are the angles subtended by AB, BC respectively, then

(1) β is the supplement of 2α.

(2) $12\cos\alpha = 7\,2\operatorname{vers}\beta = 1$.

14. The angle of elevation of the top of a mountain from a point at its base is observed to be γ. A path leads directly from that point to the top, being for some distance inclined to the horizon at an angle α and for the remainder of the distance at a greater angle β. At the point where the path becomes steeper, the vertical altitude is found by the barometer to be a. Show that the height of the mountain is

$$\frac{a\,(\cot\alpha - \cot\beta)}{\cot\gamma - \cot\beta}.$$

15. A vertical stick casts a shadow of length b from a lamp upon a horizontal plane. The horizontal and vertical distances of the bottom of the stick from its shadow are a and c respectively. If the stick subtends equal angles at the two ends of its shadow, show that the height of the lamp is

$$\frac{abc}{c^2 - a^2}.$$

16. An object is observed at three points A, B, C lying in a horizontal line which passes directly underneath the object. The angles of elevation at A, B, C are α, 2α, 3α respectively. If $AB = a$, $BC = b$, show that the height of the object is

$$\frac{a}{2b}\sqrt{\{(a+b)\,(3b-a)\}}.$$

If $\tan\alpha = \frac{2}{9}$, show that $13a = 23b$.

17. From a point on a hill-side of constant inclination, the altitude of the highest point of an obelisk at the top of the hill is observed to be α: a feet nearer the top of the hill it is β. Show that if θ be the inclination of the hill, the height of the obelisk is

$$\frac{a \sin(\alpha - \theta) \sin(\beta - \theta)}{\sin(\beta - \alpha) \cos\theta}.$$

18. A and B are 50 yards apart, C midway between them. From a point D, it is observed that AC subtends an angle of 60° and CB an angle of 30°. Find the distance BD.

19. A vertical tower c ft. in height stands on the top of a hill. A and B are two points in the same horizontal plane at the foot of the hill. From A the tower lies due north and the angular elevation of its base and top are α and β. From B the tower lies due west and the angular elevation of its top is 45°. Show that

$$AB = \frac{c \cos\alpha}{\sin(\beta - \alpha)}.$$

20. A is the top of a tower, B and C are points in the horizontal plane on which it stands. The elevations of A at B and C are β and γ respectively; and the angle subtended at A by BC is α. Show that

$$\cot ABC = \sin\gamma \operatorname{cosec}\alpha \operatorname{cosec}\beta - \cot\alpha.$$
$$\cot ACB = \sin\beta \operatorname{cosec}\alpha \operatorname{cosec}\gamma - \cot\alpha.$$

21. A person walking along a straight road observes that the greatest angle which a building subtends is α. From this point he walks a distance c, and the front of the building is now just along his line of sight making an angle β with the road. Show that the length of the building is

$$c \sin\alpha \sin\beta \sec\tfrac{1}{2}(\beta + \alpha) \sec\tfrac{1}{2}(\beta - \alpha).$$

22. A man, standing at a point close to the side of a base of a pyramid whose base is square sees the sun disappear over an edge of the pyramid, half-way along it. Show that, if a and b are the distances of the man from the two ends of the side at

which he is standing, and θ the altitude of the sun, the height of the pyramid will be

$$\tfrac{1}{2} \tan \theta \sqrt{(10a^2 - 4ab + 2b^2)}.$$

23. A, B, C are three points in a horizontal plane forming an isosceles triangle right-angled at C, and $AB = c$. At A, B, C the angles of elevation of an object P are α, α, β respectively. Find h and show that β cannot be less than the angle whose tangent is $\dfrac{\tan \alpha}{\sqrt{2}}$.

24. Three mountain peaks A, B, C appear to the observer to be in a straight line, when he stands at each of two places P and Q in the same horizontal line. The angle subtended by AB and BC at each place is α; and the angles AQP, CPQ are ϕ and ψ.

Prove that the heights of the mountains are as

$\cot 2\alpha + \cot \psi : \tfrac{1}{2} \tan \alpha (\cot \alpha + \cot \psi)(\cot \alpha + \cot \phi) : \cot 2\alpha + \cot \phi$;

and that if QB cut AC in D,

$$AC = CD \times \sin 2\alpha (\cot \alpha + \cot \psi).$$

25. A man standing on a plane observes a row of equal and equidistant pillars, the 10th and 17th of which subtend the same angle as they would if they were in the position of the first and were respectively $\frac{1}{2}$ and $\frac{1}{3}$ of the height. Show that, neglecting the height of the eye, the line of the pillars is inclined to the line drawn to the first at an angle whose secant is nearly 2·6.

CHAPTER IX.

THE GEOMETRY OF THE TRIANGLE.

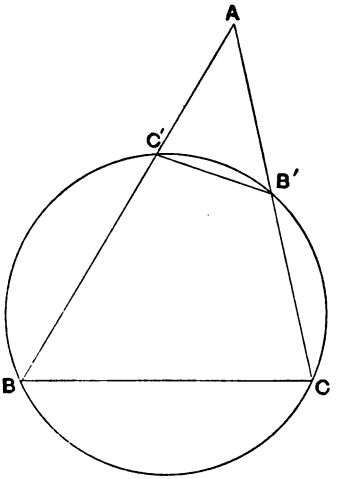
182. In this chapter will be given the most fundamental propositions on the circles and centres of a triangle.

For the sake of avoiding repetition, the following mode of lettering will be consistently adopted. (Explanation of the terms will be given in the course of the chapter.)

The triangle considered ABC. The middle points of the sides D, E, F. The feet of the perpendiculars L, M, N. The points of contact with the inscribed circle X, Y, Z. The centre of the inscribed circle, I. The centre of the circumscribed circle, S. The centre of gravity, G. The orthocentre, O. The centre of the cosine circle, K. The centre of the nine-points circle, T. The centre of the Lemoine and Brocard circles, V. The middle points of AO, BO, CO; P, Q, R. The centres of the escribed circles I_1, I_2, I_3. Their points of contact with the sides X_1, Y_1, Z_1, &c.

Antiparallels.

183. Def. Two lines are said to be **antiparallel** with respect to any angle, when the inclination of one to one of the lines containing the angle is equal and opposite to the inclination of the other to the other of the lines containing the angle.



Thus, in the figure, BC, $B'C'$ are antiparallel with respect to the angle A:—the angles ABC, $AB'C'$ being equal and opposite* to one another.

It follows that the angles $C'BC$, $C'B'C$ are together equal to two right angles; and, therefore, the quadrilateral $BCB'C'$ is inscribable in a circle. Hence antiparallels might be thus defined:—

Two opposite sides of a quadrilateral inscribable in a circle are said to be antiparallel *with respect to the angle contained by the other two opposite sides.*

The following statements are obvious:—

Lines which are, with respect to any angle, antiparallel to the same line are parallel to one another.

Through any point, one and only one line can be drawn which is, with respect to any angle, antiparallel to another line.

These statements are sufficient to indicate the analogy between antiparallels and parallels.

* That is, BC has revolved from BA in the opposite direction to that in which $B'C'$ has revolved from $B'A$.

184. Another mode of defining antiparallels is the following:

Two lines are said to be *antiparallel* with respect to two other lines when the bisectors of the angles between the first two are in the same directions as the bisectors of the angles between the second two.

Thus, in the figure, the lines BC, $B'C'$ are antiparallel with respect to the lines BC', $B'C$; for the bisector of the *obtuse* angle between BC and $B'C'$ is in the same direction as the bisector of the *acute* angle between BC' and $B'C$: and the bisector of the *acute* angle between BC and $B'C'$ is in the same direction as the bisector of the *obtuse* angle between BC' and $B'C$.

This will become clear if parallels to the four lines BC, $B'C'$, BC', $B'C$ be drawn through any point O.

It is clear that the relations between the pair BC, $B'C'$ and the pair BC', $B'C$ are *reciprocal*.

It should be observed that, if the angles ABC, ACB are equal, the antiparallels to BC with regard to A become *parallels* to BC.

185. The following proposition is important:—

A line, drawn from the vertex of an angle, cuts any two parallel intercepts of the angle in the same ratio.

This follows at once from Euc. VI. 4.

In particular:—The line, drawn from the vertex of a triangle to bisect the base, bisects all the parallels to the base.

When an antiparallel to a *side of a triangle* is spoken of, the antiparallel must be understood to be drawn *with respect to the angle opposite that side*.

Thus, in considering the triangle ABC of the figure of Art. 183, $B'C'$ is called an antiparallel *to the side* BC (without specifying the angle A).

Since the antiparallels to a side are parallel to one another, the line from the opposite angle bisecting one of them will bisect all.

Centres of Similitude.

186. DEF. The point which divides the distance between the centres of two circles internally (or externally) in the ratio of the radii of the circles is called the internal (or external) **centre of similitude** of the two circles.

PROP. The line joining the extremities of any two parallel radii of two circles cuts the line of their centres in one or other of the centres of similitude.

For, let O, o be the centres of two circles, and OR, or two parallel radii. Let Rr cut Oo in S. Then the triangles ROS, roS are equiangular;

$\therefore OS : oS = OR : or$, $\therefore$ *S is a centre of similitude.*

If Rr is a common tangent of the two circles, OR and or are both at right angles to Rr; and are, therefore, parallel: hence, *a common tangent to two circles cuts the line of their centres in one or other of the centres of similitude.*

COR. If the circles do not cut, there is a pair of common tangents through each centre of similitude, the angle between which is bisected by the line of centres.

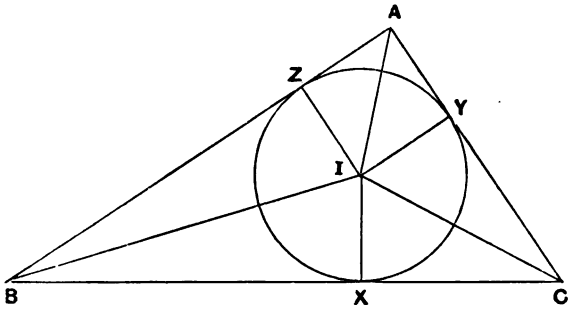
Hence *the two common tangents through one centre of similitude are* antiparallel *with respect to the angle formed by the two common tangents through the other centre of similitude.*

Pole and Polar.

187. DEF. The point of intersection of the tangents to a circle drawn from the extremities of any chord is called the **pole** of the chord: and the chord is called the **polar** of the point of intersection.

The inscribed circle.

188. PROP. I. *The internal angular bisectors of a triangle meet in a point.*



Let BI, CI—bisectors of B and C—cut in I. *AI shall bisect A.*

For, draw IX, IY, IZ perpendiculars on the sides.

Then, the triangles BXI, BZI having a common side BI and two angles of the one equal to two angles of the other, are equal in all respects, so that $IX = IZ$.

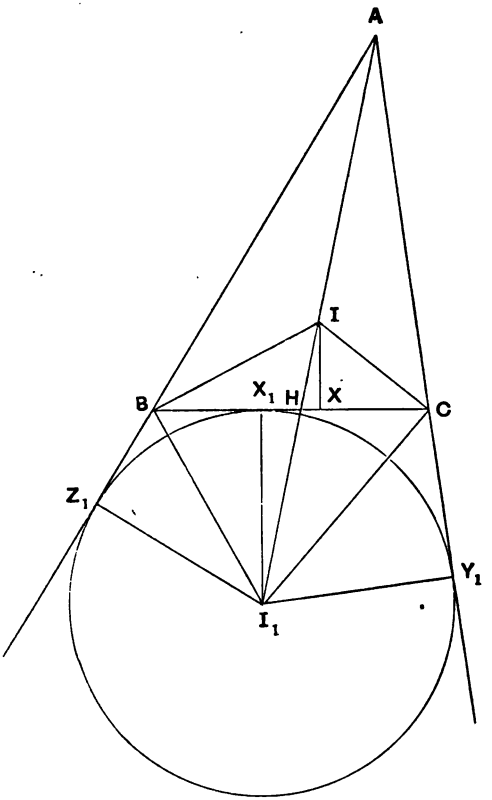
Similarly $IX = IY$, $\therefore$ $IY = IZ$.

Therefore, the triangles AZI, AYI having each a right angle and two sides of the one equal to two sides of the other, are equal in all respects, $\therefore$ *IA bisects A.*

Since IX, IY, IZ are equal and are perpendicular to the sides, $\therefore$ I is the centre of the circle of radius IX which touches the sides of ABC internally. That is, *I is the centre of the inscribed circle of ABC.*

The Escribed Circles.

189. PROP. II. *The bisectors of two external angles and the bisector of the remaining internal angle meet in a point.*



Let BI_1, CI_1—bisectors of the external angles at B and C—cut in I_1. AI_1 *shall bisect* A.

For, drawing perpendiculars I_1X_1, I_1Y_1, I_1Z_1 upon the sides, we may prove precisely as in the last proposition, that $I_1X_1 = I_1Y_1 = I_1Z_1$, and, therefore, I_1A bisects BAC.

Since the perpendiculars from I_1 on the sides are equal to one another, $\therefore$ I_1 is the centre of the circle touching AB, AC produced and BC externally. This circle is called an *Escribed Circle* of the triangle.

Cor. I. The three sides of the triangle are common tangents of the inscribed and escribed circles; and the three vertices are centres of similitude of pairs of these circles.

Thus the vertex A is the *external* centre of similitude of the inscribed circle and the A-escribed circle, and the *internal* centre of similitude of the B- and C-escribed circles.

Cor. II. If the *internal* bisector AII_1 of A cut BC in H, since HIX, HI_1X_1 are similar triangles, $IH : I_1H = IX : I_1X_1$, $\therefore$ H is the *internal* centre of similitude of the inscribed circle and the A-escribed circle.

[Similarly, if the *external* bisector I_2AI_3 of A cuts BC in H_1, H_1 is the *external* centre of similitude of the B- and C-escribed circles.]

Cor. III. The fourth common tangent of the inscribed and the A-escribed circles passes through H and is antiparallel to BC (with respect to A).

[Similarly the fourth common tangent of the B- and C-escribed circles passes through H_1 and is antiparallel to BC.]

Cor. IV. The fourth common tangent of the inscribed and the A-escribed circles cuts off from AB and AC parts, measured from A, equal to AC and AB respectively.

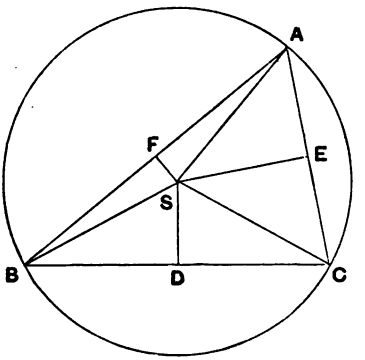
[Similarly the fourth common tangent of the B- and C-escribed circles cuts off from BA and CA produced parts, measured from A, equal to AC and AB respectively.]

Cor. V. Since BI, BI_1 bisect two adjacent supplementary angles at B, IBI_1 is a right-angle. Hence the circle on II_1 as diameter passes through B and C, and $\therefore$ the perpendiculars IX, I_1X_1 from the extremities of this diameter cut off equal segments from the chord BC;

$$\therefore BX = CX_1 \text{ and } CX = BX_1.$$

The Circumscribed Circle.

190. PROP. III. *The straight lines which bisect the sides of a triangle at right-angles meet in a point.*



Let D, E, F be the middle points of the sides of ABC.

Let FS, ES—perpendiculars on AB, AC—cut in S.

DS shall be perpendicular to BC.

For, join AS, BS, CS. Then, the triangles BFS, AFS having FS common, and $BF = AF$, and the angle BFS = the angle AFS, are equal in all respects; so that $BS = AS$.

Similarly $CS = AS$, $\therefore$ $BS = CS$.

$\therefore$ the triangles BDS, CDS having the three sides of one equal to the three sides of the other, are equal in all respects.

$\therefore$ *DS is at right-angles to BC.*

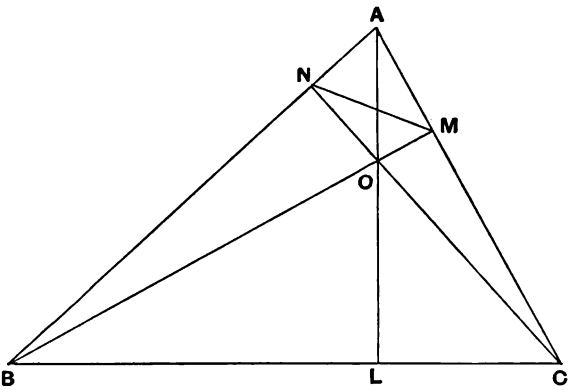
Since AS, BS, CS are equal to one another, $\therefore$ *S is the centre of the circle circumscribing ABC.*

If the triangle is obtuse-angled, the circumcentre falls *outside* the triangle, on the side remote from the obtuse angle.

COR. Since the tangent at A to the circumcircle makes with AB an angle equal to the angle BCA in the opposite segment, $\therefore$ *it is antiparallel to BC.* But the radius SA is perpendicular to the tangent at A. $\therefore$ *the lines SA, SB, SC are perpendicular respectively to the antiparallels to the sides.*

The Orthocentre.

191. PROP. IV. *The perpendiculars to the sides from the opposite angles meet in a point.*



Let BM, CN—perpendiculars on AC, AB—cut in O; and let AO cut BC in L. *OL shall be perpendicular to BC.*

For, join MN. Then, the angles ANO, AMO being right-angles, a circle goes round $AMON$; $\therefore \angle NMO = NAO$.

And the angles BNC, BMC being right-angles, a circle goes round $BNMC$; $\therefore \angle NMB = NCB$; $\therefore \angle NAO = NCB$.

$\therefore$ the triangles BAL, BCN having two angles of one equal to two angles of the other, the third angles BLA, BNC are equal.

But BNC is a right-angle, $\therefore$ *also BLA is a right-angle.*

The point O is called the Orthocentre of the triangle ABC.

If the triangle is obtuse-angled, the orthocentre falls *outside* the triangle, and within the space formed by producing through the obtuse angle its two containing sides.

COR. I. Since a circle goes round $BNMC$, $\therefore$ the sides of the triangle NML are antiparallels to the sides of the triangle ABC.

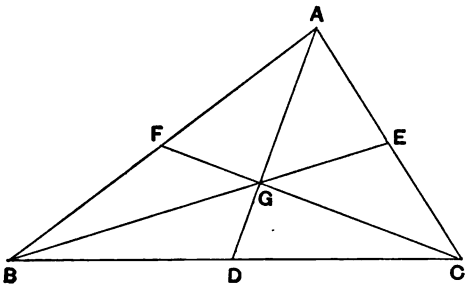
COR. II. Since angle $NMO = NAO =$ complement of B,
and angle $LMO = LCO =$ complement of B,

$\therefore$ O is the centre of the circle inscribed in LMN.

COR. III. Since AB, AC are perpendicular to ON, OM respectively, $\therefore$ A, B, C are the *escribed* centres of LMN.

The Centre of Gravity.

192. PROP. V. *The bisectors of the sides drawn from the opposite angles meet in a point.*



Let D, E, F be the middle points of the sides of ABC. Let BE, CF cut in G. *AG, GD shall be in one straight line.*

For, because $BF = AF$,

$\therefore$ the triangles BCF, ACF are equal; and also the triangles BGF, AGF are equal, $\therefore$ the remainders BGC, AGC are equal.

Similarly, BGC, AGB are equal, $\therefore$ $AGB = AGC$.

And because $BD = CD$, $\therefore$ $BGD = CGD$.

$\therefore$ AGB, BGD are together equal to AGC, CGD.

That is, the lines AG, GD bisect the triangle ABC.

But the straight line AD bisects the triangle ABC,

$\therefore$ the lines AG, GD coincide with the straight line AD.

The point G is the Centre of Gravity of the triangle ABC.

COR. I. Joining DE, $\because$ $AE : EC = BD : DC$, $\therefore$ DE is parallel to AB. $\therefore$ the triangles ABC, EDC are similar,

$$\therefore AB : DE = BC : DC.$$ But BC = twice DC.

$\therefore$ *the sides of the triangle DEF are half those of ABC, and its angles are equal to those of ABC.*

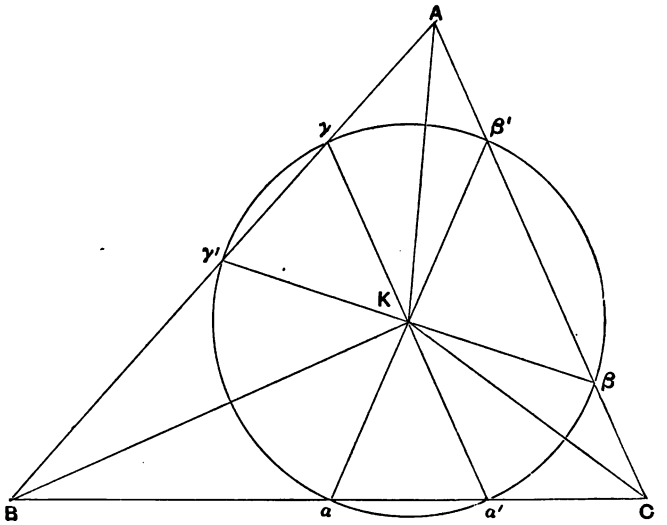
COR. II. Also the triangles AGB, GDE are similar,

$$\therefore AG : GD = AB : DE.$$

$\therefore$ AG = twice GD: or, *G trisects the distances AD, BE, CF.*

The Cosine Circle.

193. PROP. VI. *The bisectors of the antiparallels to the sides drawn from the opposite angles meet in a point.*



Let BK, CK—bisectors of the antiparallels to AC and AB—cut in K. *AK shall bisect the antiparallels to BC.*

Through K draw $\beta K\gamma'$, $\gamma K\alpha'$, $\alpha K\beta'$ antiparallel to BC, CA, AB respectively; so that the angles at α and at α' equal A; those at β and at β' equal B; and those at γ and at γ' equal C.

$\therefore K\alpha = K\alpha'$; $K\beta = K\beta'$; and $K\gamma = K\gamma'$.

But $\because$ BK bisects the antiparallels to CA, $\therefore K\gamma = K\alpha'$.

And $\because$ CK bisects the antiparallels to AB, $\therefore K\alpha = K\beta'$.

$\therefore$ by above, $K\gamma = K\beta'$ and $K\beta = K\gamma'$.

$\therefore$ *AK bisects the antiparallels to BC.*

Since $Ka = Ka' = K\beta = K\beta' = K\gamma = K\gamma'$, $\therefore$ the circle with centre K and radius Ka passes through α, α', β, β', γ, γ'. This is known as the *Cosine Circle.* It is a unique circle, as regards the triangle, in possessing the property of cutting the sides *at the extremities of three diameters.*

The construction shows that the quadrilaterals $\alpha\alpha'\beta'\gamma$, $\beta\beta'\gamma'\alpha$, $\gamma\gamma'\alpha'\beta$ are *rectangles.*

Also the triangles $\beta\gamma\alpha$ and $\gamma'\alpha'\beta'$ having their sides perpendicular to those of ABC are equiangular to ABC.

Cor. *The perpendiculars from K upon the sides are proportional to those sides.*

For if Ky, Kz are the perpendiculars upon AC, AB respectively, since the angles at β and γ are equal to B and C respectively,

$\therefore$ the triangles $K\beta y$, $K\gamma z$ are similar to the triangles ABL, ACL (of Prop. IV.), respectively.

$\therefore Ky : AL = K\beta : AB$ and $Kz : AL = K\gamma : AC$.

$\therefore Ky \,.\, AB = AL \,.\, K\beta = AL \,.\, K\gamma = Kz \,.\, AC$.

$\therefore Kx : Ky : Kz = BC : CA : AB$.

The Ex-Cosine Circles.

194. Let $K_1K_2K_3$ be the poles of the sides of ABC with respect to the circumcircle. Then, by Prop. III. Cor., the tangents BK_1, CK_1 are antiparallels to AC, AB respectively.

Through K_1 draw $\beta_1K_1\gamma_1$ antiparallel to BC.

Then we may prove precisely as in the last proposition that $K_1\beta_1 = K_1B = K_1C = K_1\gamma_1$; $\therefore$ *the circle with centre K_1 and radius K_1B or K_1C passes through β_1, γ_1.*

This circle may be called an *Ex-Cosine Circle.*

Moreover K_1, being the middle point of $\beta_1\gamma_1$, lies on AK.

Hence *K is the intersection of the lines joining the vertices of the triangle to the poles of the opposite sides with respect to the circumcircle.*

Cor. The perpendiculars from K_1, from K_2, and from K_3 upon the sides are proportional to the sides.

Relations between S, O, G, K.

195. PROP. VII. *The bisector of the angle* A *is also the bisector of each of the angles* SAO *and* GAK.

For, in Prop. III., BSA at the centre is twice C at the circumference opposite AB. $\therefore FSA = C$. $\therefore FAS$ is the complement of $C = CAL$, in Prop. IV.

i.e. the angle that AS makes with AB is equal to the angle that AO makes with AC.

$\therefore$ *the bisector of* BAC *is also the bisector of* SAO.

Again, in Prop. VI., $A\beta\gamma'$ is similar to ABC; and since βK is half $\beta\gamma'$, and BD, in Prop. V., is half BC, $\therefore$ the triangles $A\beta K$, ABD are similar. $\therefore$ the angle BAD = the angle CAK.

i.e. the angle that AG makes with AB is equal to the angle that AK makes with AC.

$\therefore$ *the bisector of* BAC *is also the bisector of* GAK.

The results of Props. I.—VII. may be thus summarised:—

Consider the lines from the vertex A to the several centres.

AI_1 bisects the *internal angle* at A,

I_2AI_3 bisects the *external angle* at A;

and hence AI_1 and I_2AI_3 are perpendicular to one another.

AS is perpendicular to the *antiparallels* to BC,

AO is perpendicular to the *parallels* to BC;

and hence AS and AO are antiparallel* with respect to A.

AG bisects the *parallels* to BC,

AK bisects the *antiparallels* to BC;

and hence AG and AK are antiparallel* with respect to A.

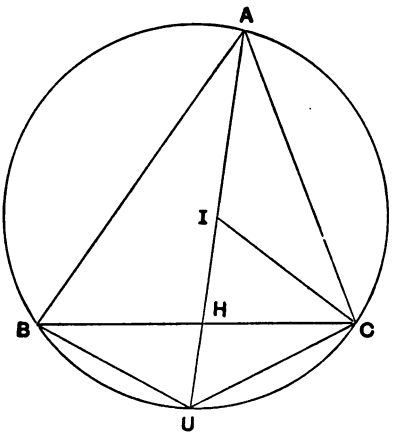
It may be shown generally that if any three lines through the vertices of a triangle are concurrent, their antiparallels through the vertices are concurrent. [See Prop. XVI.]

The two points of concurrence may be called *anticentres*. Thus I is its own anticentre: O is the anticentre to S; and K is the anticentre to G.

* Antiparallels, with respect to an angle, when drawn through the angle, are called *Isogonal Conjugates*. The definition by means of the quadrilateral inscribable in a circle here fails.

Relations between the Incircle and the Circumcircle.

196. PROP. VIII. *If* AI *be produced to cut* BC *in* H *and the circumcircle in* U, *then* UI = UC = UB = *the mean proportional between* UH *and* UA.



For the chords *BU*, *CU* of the circumcircle, subtending equal angles *BAU*, *CAU* at the circumference are equal.

And, angles in the same segment being equal, $BCU = BAU = CAU$.

And, the angle at *U* being common, the triangles *HCU*, *CAU* are similar, $\therefore$ $UH : UC = UC : UA$.

Again, join *CI*.

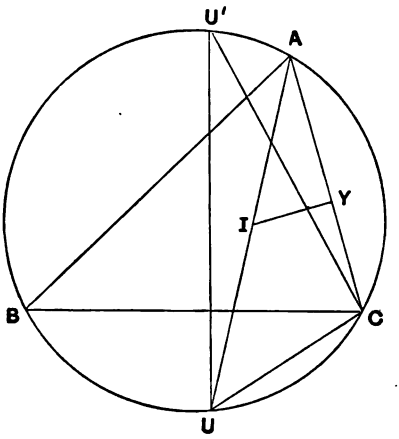
Then $UIC = IAC + ICA = HCU + ICH = UCI$.

$\therefore$ $UC = UI$ = the mean proportional between *UH* and *UA*.

COR. Let *UD*, *IX*, *AL* be drawn perpendicular to *BC*. Then *D* is the middle point of *BC*, *X* its point of contact with the incircle, and *L* the foot of the perpendicular on it from *A*.

Now the triangles *DUH*, *XIH*, *LAH* are similar; $\therefore$ from the proposition that *UI* is the mean proportional between *UH* and *UA*, it clearly follows that *DX is the mean proportional between DH and DL*.

197. PROP. IX. *The rectangle of the segments of any chord of the circumcircle drawn through the centre of the incircle = twice the rectangle of their radii.*



For draw the diameter UU' of the circumcircle perpendicular to BC: and the radius IY of the incircle perpendicular to AC. Join CU'.

Then, by Prop. VIII., U, I, A are in a straight line, and $UI = UC$.

Also, UCU' in a semicircle = the right-angle IYA, and $UU'C$, UAC in the same segment are equal.

$\therefore$ the triangles UCU', IYA are similar.

$\therefore$ $UC : UU'$, i.e. $UI : UU' = IY : IA$.

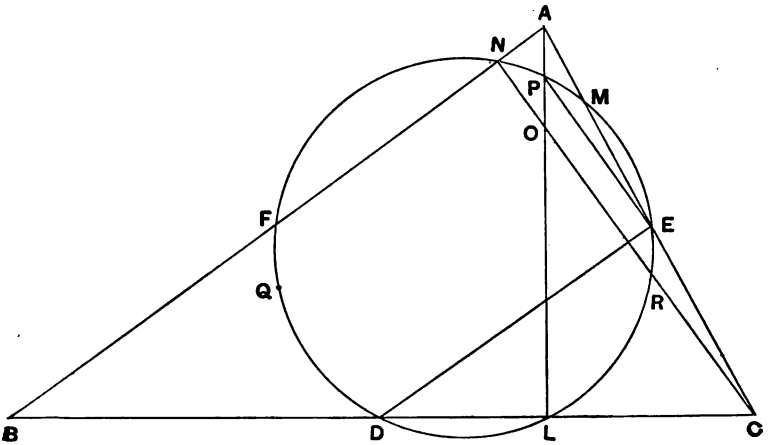
$\therefore$ rectangle UI, IA = rectangle UU', IY.

i.e. *the rectangle of the segments* UI, IA = *twice the rectangle of the radii of the two circles.*

$\therefore$ *the rectangle of the segments of* any *chord of the circumcircle drawn through* I *is equal to twice the rectangle of the radii.*

The Nine-Points Circle.

198. PROP. X. *The middle points of the sides, the feet of the perpendiculars from the opposite angles, and the points midway between the angles and the orthocentre lie on a circle.*



Let D, E, F be the middle points of the sides of ABC; AL, BM, CN the perpendiculars from A, B, C cutting in O; P, Q, R the middle points of AO, BO, CO. *The nine points* D, E, F; L, M, N; P, Q, R *will lie on a circle.*

Draw the lines AL, CN; PE, DE.

$\because$ DE bisects BC and AC, $\therefore$ DE is parallel to AB. And

$\because$ PE bisects AO and AC, $\therefore$ PE is parallel to OC.

But AB and OC are at right-angles to one another;

$\therefore$ also DE and PE are at right-angles to one another.

$\therefore$ the circle on DP as diameter passes through E.

For the same reason this circle passes through F.

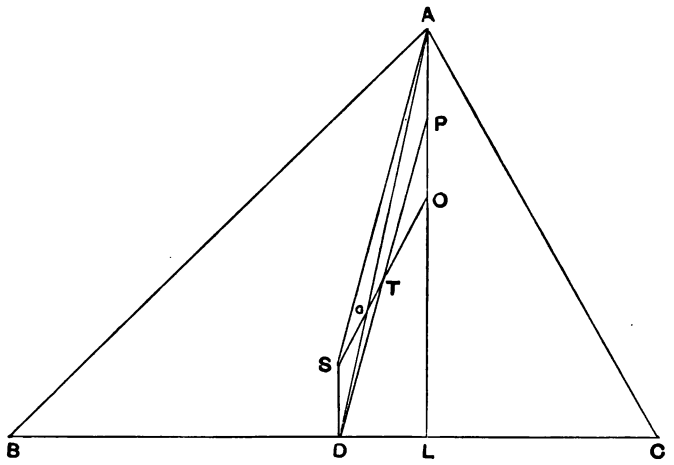
And it passes through L, because DLP is a right-angle.

$\therefore$ the circle circumscribing DEF passes through P and L.

Similarly it passes through Q, M; R, and N.

Hence *the nine-points* DEF, PQR, LMN *lie on a circle.*

199. PROP. XI. *The gravity centre trisects and the nine-points centre bisects the line joining the circumcentre and orthocentre.*



Let T the middle point of DP be the nine-points centre.

(1) Produce OT to S. Let $TS = TO$. *S shall be the circumcentre.*

Join SD. Then the triangles STD, OTP having two sides and an included angle at T equal, are equal in all respects.

$\therefore$ SD is parallel to OP and $\therefore$ at right-angles to BC.

Similarly SE, SF are at right-angles to AC, AB.

$\therefore$ *S is the circumcentre.*

(2) Join AD cutting ST in G. *G shall be the gravity centre.*

For the triangles SGD, OGA are similar, and

$$AO = 2 \,.\, PO = 2 \,.\, SD, \quad \therefore \; AG = 2 \,.\, GD.$$

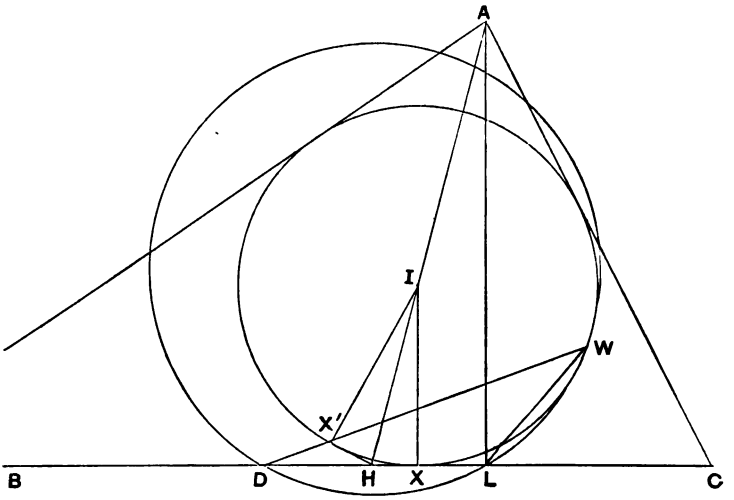
$\therefore$ *G is the gravity centre.*

Also $GO = 2 \,.\, GS$, $\therefore$ *G trisects the distance SO.*

COR. I. The radius of N.P.C. $= \frac{1}{2}$ that of the circumcircle.

COR. II. The angle DPL, which is the angle in the segment of the nine-points circle opposite DL, is equal to SAL; and is therefore, by Prop. VII., double of the angle between the bisector of A and the perpendicular to BC.

200. PROP. XII. *The nine-points circle touches the inscribed circle.*



Let D bisect BC, AIH bisect A, and let IX and AL be perpendiculars upon BC.

Draw HX' to touch the incircle at X'. Through D draw $DX'W$ to cut the incircle in W. Join WL.

Then (by Prop. VIII., Cor.), $DH \cdot DL = DX'^2$.

$\therefore$ $DH \cdot DL = DX' \cdot DW$ or $DH : DX' = DW : DL$.

$\therefore$ the triangles DWL, DHX' are similar.

$\therefore$ the angle $DWL = DHX' = XIX' = 2 \cdot HIX =$ (by Prop. XI., Cor. II.) angle in the segment of nine-points circle opposite DL.

$\therefore$ (1) *W is on the nine-points circle.*

Also the tangent at W to the nine-points circle makes with WX' an angle equal to $DLW = DX'H =$ angle in the segment of the incircle opposite $X'W$.

$\therefore$ (2) the tangents at W to the incircle and to the nine-points circle coincide.

$\therefore$ *the incircle and the nine-points circle touch at W.*

201. In the same way it may be shown that the nine-points circle touches each of the escribed circles.

For since $BX_1 = CX$, and $BD = CD$, $\therefore$ the remainder

$$DX_1 = DX. \quad \therefore DX_1^2 = DH \cdot DL.$$

Moreover the escribed centre I_1 is on AH.

Hence if the tangent through H is drawn to the escribed circle touching it at X_1', and a chord is drawn through DX_1', the other end of the chord will be the point of contact with the nine-points circle.

Now H being the centre of similitude of the inscribed and escribed circles, the tangent through H to either circle is a *common* tangent. Hence we see that

If the fourth common tangent to the inscribed and an escribed circle be drawn, and through the points of contact chords of these circles be drawn to pass through the middle point of the side, the other extremities of the chords will be the points of contact with the nine-points circle.

The Lemoine Circle.

202. PROP. XIII. *The parallels to the sides drawn through the centre of the cosine circle cut the sides in six points which lie on a circle.*

Construction. Let V be the middle point of KS; and α the middle point of AK. Through α draw A_1A_2 antiparallel to BC. Join $V\alpha$, VA_2.

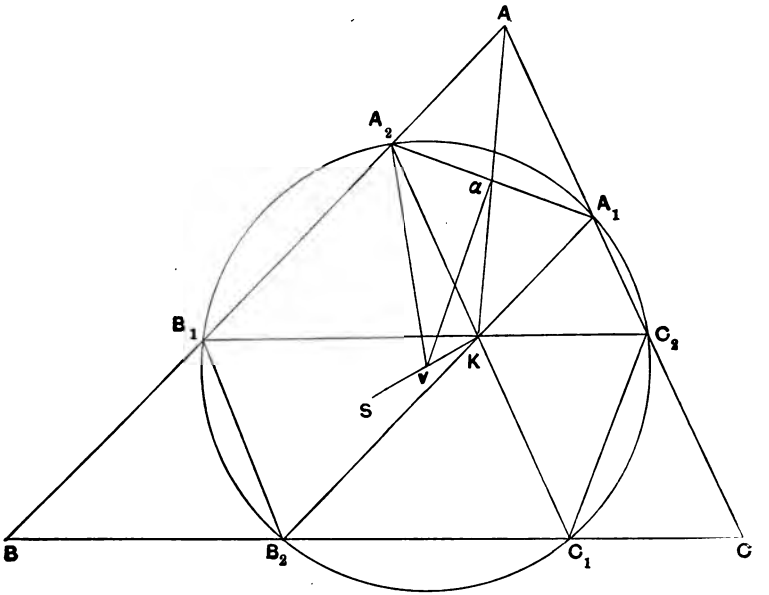
Then $\because$ V and α are the middle points of SK and AK,

$\therefore$ $V\alpha$ is parallel to SA and is half SA.

But SA is perpendicular to the antiparallels to BC and is equal to the circumradius.

$\therefore$ $V\alpha$ is perpendicular to A_1A_2 and equal to half the circumradius.

Again, drawing the antiparallel to BC from K to AB, we have a line parallel to $A_2\alpha$ and equal to the cosine-radius.



$\therefore$ $A_2\alpha$ is half the cosine-radius.

$\therefore$ $VA_2^2 =$ (half circumradius)^2 + (half cosine-radius)^2.

$\therefore$ if B_1B_2, C_1C_2 are similarly drawn, then

$$VA_2 = VA_1 = VB_2 = VB_1 = VC_2 = VC_1.$$

$\therefore$ the circle with centre V and radius VA_2 passes through $A_1A_2B_1B_2C_1C_2$.

Also, since AK, A_1A_2 bisect one another, A_1AA_2K is a parallelogram.

$\therefore$ B_1C_2, C_1A_2, A_1B_2 are parallels through K to the sides.

This circle is unique, as regards the triangle, in possessing the property of cutting the sides *in three concurrent chords parallel to the sides.* It is called the Lemoine circle.

203. If, instead of *bisecting* SK and AK, we divide SK and AK in any *the same* ratio at two new points V and α, and if the construction of the last article be made, then, by the same reasoning, the six new points $A_1A_2B_1B_2C_1C_2$ will lie on a circle.

If, further, K_a, K_b, K_c are taken on AK, BK, CK at distances from A, B, C equal to twice $A\alpha$, $B\beta$, $C\gamma$ respectively, then the figures $A_1AA_2K_a$, $B_1BB_2K_b$, $C_1CC_2K_c$ will be parallelograms. And because K_a, K_b, K_c divide KA, KB, KC in the same ratio, therefore the lines K_bK_c, K_cK_a, K_aK_b, are parallel respectively to BC, CA, AB.

Hence we have the proposition:

Three parallels to the sides, drawn so as to cut one another in three points K_a, K_b, K_c *on the lines* KA, KB, KC, *will cut the sides in six points lying on a circle.*

The circles so described are called *Tucker's Circles.* They are unique in cutting the sides in six points, *which, being joined in pairs, form three chords parallel to the sides.*

Observe that the triangles $A_1B_1C_1$, $A_2B_2C_2$ are similar to ABC. For $\angle B_2A_2C_2 = \angle B_2A_1C_2 = A$; and so on.

All Tucker's circles have their centres in SK.

In particular;

Taking the centre at K, we have the *cosine circle.*

Taking the centre at S, we have the *circumcircle.*

Taking the centre at the middle point of SK, we have Lemoine's circle.

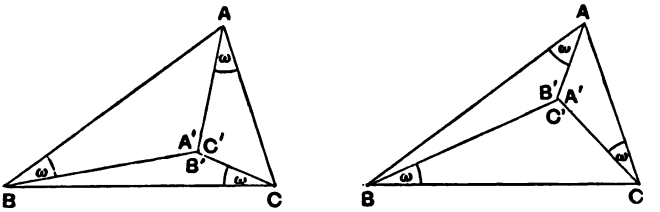
The Brocard Points and the Brocard Angle.

204. PROP. XIV. *To draw from the vertices of a triangle three lines intersecting in a point, which shall subtend equal angles at the other vertices.*

On AB, BC, CA describe segments of circles containing angles equal to A', B', C' (the supplements of A, B, C) respectively. (Fig. 1.)

These segments will cut in one point, Ω say, because the angles A', B', C' are together equal to four right angles.

Then $\because$ the exterior angle at Ω of the triangle $B\Omega C = B$,
$\therefore$ the two interior angles $CB\Omega$ and $BC\Omega$ are together equal to B.
$\therefore$ the angles $BC\Omega$, $AB\Omega$ are equal.



Thus, *the three angles* $AB\Omega$, $BC\Omega$, $CA\Omega$ *are equal*.....(1).

Again on CA, AB, BC describe segments of circles containing angles equal to A', B', C' respectively, cutting in Ω'. (Fig. 2.) Then, as before,

the three angles $AC\Omega'$, $BA\Omega'$, $CB\Omega'$ *are equal*.....(2).

If possible, let the angles of (1) and of (2) be unequal.

Suppose those of (1) are the greater.

Then from the triangles $A\Omega B$, $A\Omega' C$, containing the angle A',

$$A\Omega : B\Omega > A\Omega' : C\Omega'.$$

Similarly $B\Omega : C\Omega > B\Omega' : A\Omega'$, and $C\Omega : A\Omega > C\Omega' : B\Omega'$.

But compounding the last two disproportions, we have $B\Omega : A\Omega > C\Omega' : A\Omega'$, which contradicts the first disproportion.

$\therefore$ the six angles of (1) and (2) are equal.

The value, ω say, of each of these angles is called the *Brocard Angle:* and the points Ω, Ω', the *Brocard Points.* These points are obviously anticentres.

Cor. Turning the above disproportions into proportions, we see that the following pairs of triangles are *similar*, viz.:—

$A\Omega B$ and $A\Omega' C$; $B\Omega C$ and $B\Omega' A$; $C\Omega A$ and $C\Omega' B$.

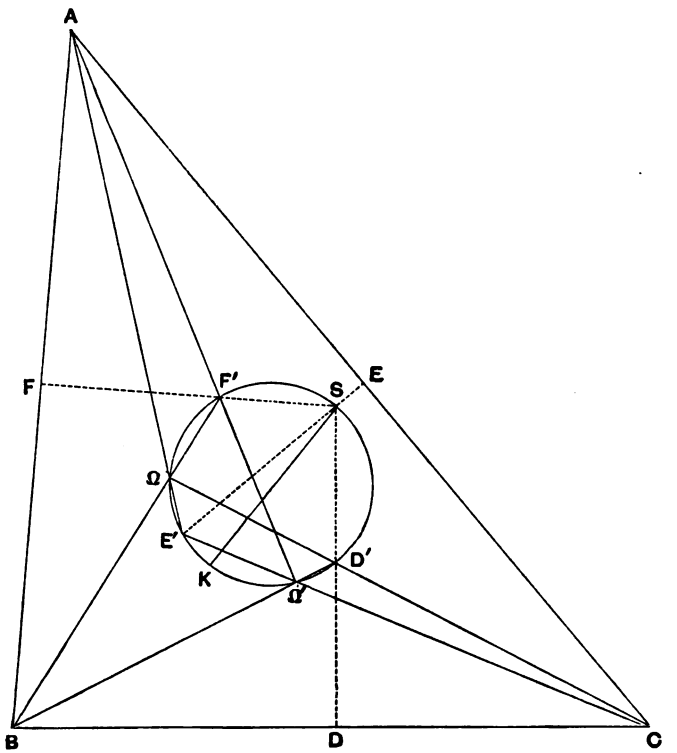
Also since $AB : A\Omega = AC : A\Omega'$, $\therefore$ the sides about the equal angles $BA\Omega'$, $CA\Omega$ of the triangles $A\Omega' B$, $A\Omega C$ are reciprocally proportional:

Thus the following pairs of triangles are *equal*, viz.:—

$A\Omega' B$ and $A\Omega C$; $B\Omega' C$ and $B\Omega A$; $C\Omega' A$ and $C\Omega B$.

The Brocard Circle.

205. PROP. XV. *The Brocard Points, and three intersections of the Brocard Lines lie on the circle whose diameter is the line joining the centres of the Circumcircle and the Cosine Circle.*



Let $B\Omega'$, $C\Omega$ cut in D'; $C\Omega'$, $A\Omega$ cut in E'; $A\Omega'$, $B\Omega$ cut in F'.

Then $A\Omega B$, $A\Omega' C$ being each the supplement of A, the angle $E'\Omega' F' = A$ and $E'\Omega F'$ is its supplement, $\therefore$ a circle goes round $E'\Omega F'\Omega'$: similarly a circle goes round $D'\Omega' E'\Omega$.

$\therefore$ the circumcircle of $D'E'F'$ passes through Ω, Ω'.....(1).

Again, $\because$ the angles $BC\Omega$, $CB\Omega'$ are equal,

$\therefore$ the perpendicular from D' on BC bisects BC (in D, say).

$\therefore$ the perpendiculars from D', E', F' on the sides cut in S the circumcentre of ABC.

And since the angle $D'SE'$ (between the perpendiculars on BC, CA) $= C$, and that $D'\Omega'E' =$ the supplement of C,

$\therefore$ *S lies on the circle* ....(2).

Lastly let the diameter through S of this circle cut it in K. Then $SD'K$, $SE'K$, $SF'K$ are right-angles.

$\therefore$ $D'K$, $E'K$, $F'K$ are parallels to BC, CA, AB.

$\therefore$ $D'D$, $E'E$, $F'F$ are equal to the perpendiculars from K on the sides.

But $\because$ the angles DBD', ECE', FAF' are equal,

$\therefore$ the triangles BDD', CEE', AFF' are similar;

$\therefore$ $DD' : EE' : FF' = BD : CE : AF = BC : CA : AB$.

$\therefore$ (Prop. VI. Cor.) *K is the centre of the cosine circle* ...(3).

Cor. I. Since SK is a diameter of the Brocard Circle, the Lemoine and Brocard Circles are *concentric.*

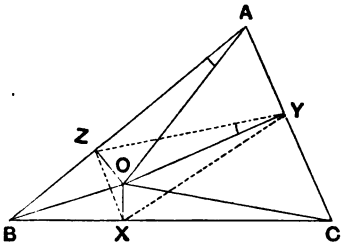
Cor. II. Since KD', KE', KF' are parallels to the sides, these meet the sides on the Lemoine Circle.

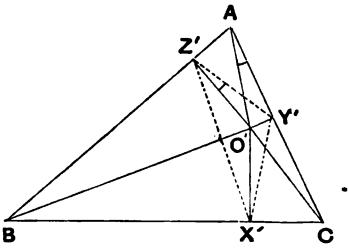
Cor. III. Since KD' is parallel to BC, $\therefore$ $\angle KD'\Omega' = \angle CB\Omega' =$ the Brocard angle. Similarly $KF'\Omega =$ the Brocard angle $\therefore$ $K\Omega$, $K\Omega'$ subtend equal angles in the Brocard circle, and $\Omega\Omega'$ *is bisected at right-angles by SK.*

Cor. IV. The triangle $D'E'F'$ is similar to ABC.

General Property of Anticentres.

206. Prop. XVI. *If three lines from the vertices of a triangle meet in a point, their antiparallels from the vertices with respect to the vertical angles will meet in a point; and the feet of the perpendiculars from the two points of concurrence on the sides will lie on a circle whose centre is the point midway between the two points of concurrence.*





Let AO, AO' be two antiparallels with respect to A, so that $\angle BAO = \angle CAO'$. On AB drop the perpendiculars OZ, $O'Z'$; and on AC, the perpendiculars OY, $O'Y'$.

Then, since a circle goes round both $AZOY$ and $AZ'O'Y'$,

$$\therefore \angle ZYO = \angle ZAO = \angle Y'AO' = \angle Y'Z'O'.$$

And the angles at O, O' are both equal to the supplement of A.

$\therefore$ the $\triangle$'s OZY, $O'Y'Z'$ are similar.

$$\therefore OZ \cdot O'Z' = OY \cdot O'Y'.$$

Similarly if BO, BO' are antiparallels with respect to B, and OX, $O'X'$ perpendiculars on BC, $OX \cdot O'X' = OZ \cdot O'Z'$,

$$\therefore OX \cdot O'X' = OY \cdot O'Y'.$$

$\therefore$ the $\triangle$'s OXY, $O'Y'X'$ are similar.

$$\therefore \angle XCO = \angle XYO = \angle Y'X'O' = \angle Y'CO'.$$

$\therefore$ *OC, $O'C$ are antiparallels with respect to C.*

Also, if T is the middle point of OO', because OX, $O'X'$ are perpendiculars on BC,

$$\therefore TX^2 = OT^2 + OX \cdot O'X'.$$

$\therefore TX = TX' = TY = TY' = TZ = TZ'$.

$\therefore$ the points $XX'YY'ZZ'$ lie on a circle whose centre is T.

In particular taking O and O' to be the orthocentre and circumcentre, we have the nine-points circle. Similar propositions hold for the gravity centre and cosine centre, and for the two Brocard Points.

Examples IX.

[The following are easy examples, which may be regarded as of the nature of book-work.]

1. If BC, $B'C'$ be antiparallel with respect to the angle at A, ABC and $AB'C'$ are similar triangles.

2. If A, B, C, D be four points *in any order* on the circumference of a circle, AB, CD shall be antiparallel with respect to the angle between AC and BD.

3. If the bisector of A cuts BC in H, the antiparallel of BC (with respect to A) through H will cut off from AB and AC parts equal to AC and AB respectively.

4. The tangent to the inscribed circle from either vertex of the triangle is equal to half the excess of the sum of the sides through the vertex over the base.

5. The tangent to the escribed circle from the opposite vertex is equal to half the sum of the sides of the triangle.

6. The external common tangent of the inscribed and escribed circles is equal to the side of the triangle, to which the escribed circle is exterior.

7. The internal common tangent of the inscribed and escribed circles is equal to the difference between the two sides to which the escribed circle is interior.

8. The fourth common tangent to the inscribed and escribed circles forms with the two sides a triangle equal in all respects to the primitive triangle.

9. The distance between the middle point of a side and its point of contact with the inscribed circle is equal to half the difference between the two other sides.

10. If the sum of two opposite sides of a quadrilateral is equal to the sum of the two other opposite sides, a circle may be inscribed in the quadrilateral.

11. The angle which each side subtends at the incentre exceeds half the opposite vertical angle by a right-angle; and that which each side subtends at an ecentre is the complement of half the opposite angle.

12. The incentre is the orthocentre of the triangle formed by the ecentres.

13. The lines joining the angle of a triangle to the extremities of that diameter of the circumcircle which bisects the base are the internal and external bisectors of the angle.

14. The distance between the centres of the inscribed and either escribed circle is bisected by the circumcircle.

15. The rectangle of the segments of any chord of the circumcircle drawn through the centre of an escribed circle is equal to twice the rectangle of their radii.

16. The feet of the perpendiculars on the sides from any point on the circumcircle lie on a straight line.

17. The lines joining the vertex to the circumcentre and orthocentre are inclined at an angle equal to the difference between the base angles.

18. If OL, the perpendicular from the orthocentre on the side BC, is produced to meet the circumcircle in O', $OL = O'L$.

19. The angles of the Pedal triangle (i.e. the triangle formed by joining the feet of the perpendiculars from the opposite angles) are supplementary respectively to the doubles of those of the primitive triangle.

20. If a triangle is formed whose sides are, with respect to the angles of ABC, antiparallels to the sides; and another formed from this in the same way, and so on, the angles of the n^{th} triangle so formed will be $60° + (-2)^n (A - 60°)$, $60° + (-2)^n (B - 60°)$, $60° + (-2)^n (C - 60°)$.

21. The angle which each side subtends at the orthocentre is the supplement of the opposite vertical angle.

22. If, in a given triangle, another triangle has to be inscribed whose sides shall be parallel respectively to those of the

given triangle, then the vertices of the inscribed triangle must be the middle points of the sides of the given triangle.

23. If, in a given triangle, another triangle has to be inscribed whose sides shall be antiparallel respectively to those of the given triangle, then the vertices of the inscribed triangle must be the feet of the perpendiculars on the sides of the given triangle.

24. The centre of gravity of the triangle DEF is the same as that of ABC, and its orthocentre is the circumcentre of ABC.

25. If the middle points of the sides of a triangle be joined, and the middle points of the sides of the triangle so formed be joined, and so on; then all the triangles so formed will have the same centre of gravity, and the circumcentre of each will be the orthocentre of the next.

26. The three triangles into which the centre of gravity divides a triangle are each one-third of the triangle.

27. The triangles cut off by joining the middle points of the sides are each one quarter of the primitive triangle.

28. The lines joining the middle points of the sides with the middle points of the perpendiculars from the opposite angles intersect at the cosine-centre.

29. The lines joining the vertices of a triangle to its points of contact with the inscribed circle intersect at the cosine-centre of the triangle formed by those points of contact.

30. The triangles into which the cosine-centre divides a triangle are to one another as the squares on the sides.

31. The line joining the vertex of the triangle to the cosine-centre divides the base into segments which are to one another as the squares on the other sides.

32. The radius of the cosine-circle is to that of the circum-circle as the perpendicular from the cosine-centre on either side is to half that side.

33. The distance of a vertex from the cosine-centre is to its distance from the middle point of the opposite side as the cosine-diameter is to the opposite side.

34. The cosine-centre is the centre of gravity of the triangle formed by the feet of the perpendiculars from it upon the sides.

35. The line joining the circumcentre of the three ecentres to the incentre of a triangle is bisected by the circumcentre.

36. If C' be taken on AB so that $AC' = AC$, and if the bisector of A cut CC' in D' and BC in H, then DD' is the mean proportional between DH and DL.

Show how, from the last example, it follows that the nine-points circle and inscribed circle touch.

37. If B_1C_2, C_1A_2, A_1B_2 be any three diameters of a circle and a triangle be formed by joining the extremities A_1A_2, B_1B_2, C_1C_2, then the circle is the cosine-circle of the triangle so formed.

38. Show that, by the method of the last example, eight different triangles may be described with the same three diameters: and that, taking any two of these triangles, either all the sides of the one are parallel or one is parallel and the other two perpendicular to the sides of the other.

39. In Lemoine's circle the three chords of intersection opposite to the three parallels to the sides are antiparallels to the sides and each is equal to the cosine-radius.

40. Each of the six triangles into which the Lemoine Hexagon is divided by joining its vertices to the cosine-centre is similar to the primitive triangle.

41. The three segments into which the Lemoine circle cuts each side are to one another as the squares on the three sides.

42. The circles, whose intersection gives one of the Brocard Points, may be defined as passing through one vertex and touching the opposite side at one of its extremities.

43. If any points α, β, γ be taken on the sides BC, CA, AB of a triangle, the circumcircles of $A\beta\gamma$, $B\gamma\alpha$, $C\alpha\beta$ will cut in a single point.

44. Let Ω be the Brocard Point at which AC subtends the supplement of A; and let α, β, γ be such points taken in BC, CA, AB that the circumcircles of $A\beta\gamma$, $B\gamma\alpha$, $C\alpha\beta$ cut in Ω*. Then (1) $\alpha\beta\gamma$ is similar to BCA. (2) Ω is one of the Brocard Points of $\alpha\beta\gamma$. (3) The Brocard angle of $\alpha\beta\gamma$ is equal to that of BCA. (4) The angles $B\Omega\alpha$, $C\Omega\beta$, $A\Omega\gamma$ are equal (being the angles by which the figure BCA would have to rotate round Ω in order to take up the position of the similar figure $\alpha\beta\gamma$).

45. The cosine-centre is a Brocard Point of each of the triangles, similar to the primitive triangle, in which the Lemoine Circle cuts the sides:—the other Brocard Points of these triangles being the Brocard Points of the primitive triangle.

46. If X, Y are the feet of the perpendiculars from O on AB, AC; then XY is perpendicular to the antiparallel of AO with respect to BAC.

47. The centre of the circle which passes through the feet of the perpendiculars from the Brocard Points on the sides is the point of intersection of the line joining the Brocard Points with the line joining the circumcentre and cosine-centre.

48. The Brocard Circle passes through the middle points of the three chords of the circumcircle which join the cosine-centre to the vertices.

* Thus: take α anywhere in BC, and let the circles $\Omega B\alpha$, $\Omega C\alpha$ cut AB, AC respectively in γ, β.

CHAPTER X.

TRIGONOMETRICAL FORMULÆ FOR CIRCLES AND RECTILINEAL FIGURES.

§ 1. Circles and Triangles.

207. *Expressions for the* area (S) *of a triangle.*

By Art. 117 the area of a triangle = *half the product of two sides and the sine of the non-obtuse angle included by them.*

Now since the sine of an angle is equal to the sine of its supplement (Art. 137), the sine of the *internal* angle may be always used in this expression.

Thus: *given two sides* (a, b) *and the included angle* (C),

$$S \equiv \text{area of triangle} = \tfrac{1}{2}ab \sin C \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now $$a = \frac{c \sin A}{\sin C} \text{ and } b = \frac{c \sin B}{\sin C},$$

$\therefore$ substituting in (1),

$$S \equiv \text{area of triangle} = \frac{c^2 \sin A \sin B}{2 \sin C} \ldots\ldots\ldots\ldots (2).$$

This form gives the area in terms of the angles and one side.

Again, by Arts. 150, 152, (1) becomes

$$S \equiv \text{area of triangle} = \sqrt{\{s(s-a)(s-b)(s-c)\}} \ldots\ldots (3).$$

This form gives the area in terms of three sides.

208. *Expressions for the* radius (r) *of the* inscribed *circle.*

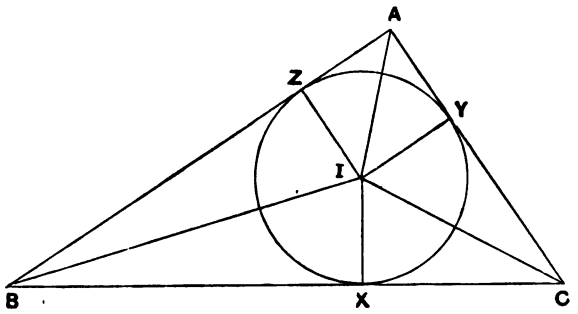
Bisect the angles at B and C by BI and CI.

Draw perpendiculars IX, IY, IZ on the sides.

Then $IX = IY = IZ = r$. Now

$$\text{area of } \triangle BIC = \tfrac{1}{2}BC \,.\, IX = \tfrac{1}{2}a \,.\, r,$$
$$\text{area of } \triangle CIA = \tfrac{1}{2}CA \,.\, IY = \tfrac{1}{2}b \,.\, r,$$
$$\text{area of } \triangle AIB = \tfrac{1}{2}AB \,.\, IZ = \tfrac{1}{2}c \,.\, r,$$

$\therefore$ (adding) $$\text{area of } \triangle ABC = \tfrac{1}{2}(a+b+c)\, r = s \,.\, r,$$



$$\therefore\ r = \frac{S}{s} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now $AY = AZ;\ BZ = BX;\ CX = CY.$

$$\therefore\ AY + BX + CX = \tfrac{1}{2}(a+b+c) = s.$$

$\therefore\ AY$ or $AZ = s - a;\ BZ$ or $BX = s - b;\ CX$ or $CY = s - c.$

$$\therefore\ r = IY = AY \tan IAY = (s-a)\tan \tfrac{1}{2}A \ldots\ldots\ldots (2).$$

Or again:

$$r = IX = IC \sin ICX = \frac{BC \sin IBC}{\sin BIC} \cdot \sin ICX$$

$$= \frac{a \sin \frac{1}{2}B \sin \frac{1}{2}C}{\sin \frac{1}{2}(B+C)} = \frac{a \sin \frac{1}{2}B \sin \frac{1}{2}C}{\cos \frac{1}{2}A} \ldots\ldots\ldots\ldots (3).$$

209. *Expressions for the* radius (r_1) *of the* escribed *circle.*

Bisect the exterior angles at B and C by BI_1, CI_1.

Draw the perpendiculars I_1X_1, I_1Y_1, I_1Z_1 on the sides.

Then $I_1X_1 = I_1Y_1 = I_1Z_1 = r_1$. Then

$$\text{area of } \triangle BI_1C = \tfrac{1}{2}BC \,.\, I_1X_1 = \tfrac{1}{2}a \,.\, r_1,$$

$$\text{area of } \triangle CI_1A = \tfrac{1}{2}CA \,.\, I_1Y_1 = \tfrac{1}{2}b \,.\, r_1,$$

$$\text{area of } \triangle AI_1B = \tfrac{1}{2}AB \,.\, I_1Z_1 = \tfrac{1}{2}c \,.\, r_1,$$

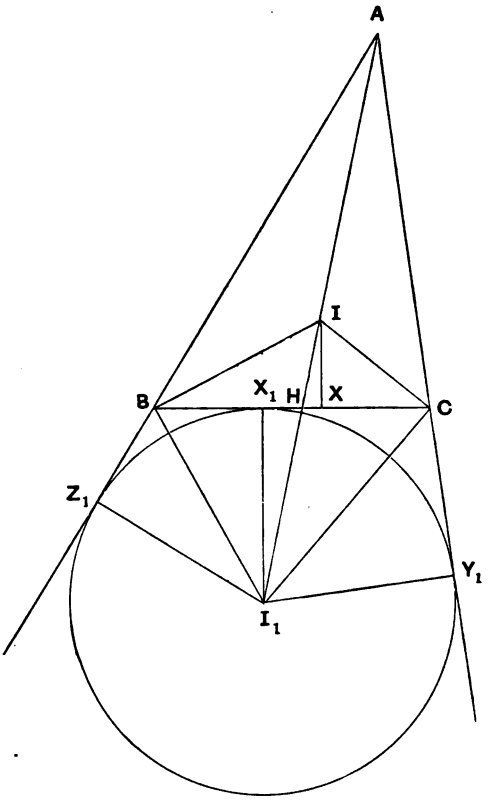
$\therefore$ (subtracting BI_1C from $CI_1A + AI_1B$),

$$\text{area of } \triangle ABC = \tfrac{1}{2}(b+c-a)\, r_1 = (s-a)\, r_1,$$

$$\therefore\ r_1 = \frac{S}{s-a} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now $AY_1 = AZ_1;\ BZ_1 = BX_1;\ CX_1 = CY_1.$

$\therefore\ AZ_1 = AB + BX_1$ and $AY_1 = AC + CX_1.$



$$\therefore\ AY_1 \text{ or } AZ_1 = \tfrac{1}{2}(a+b+c) = s,$$

and BX_1 or $BZ_1 = s-c$; CX_1 or $CY_1 = s-b$.

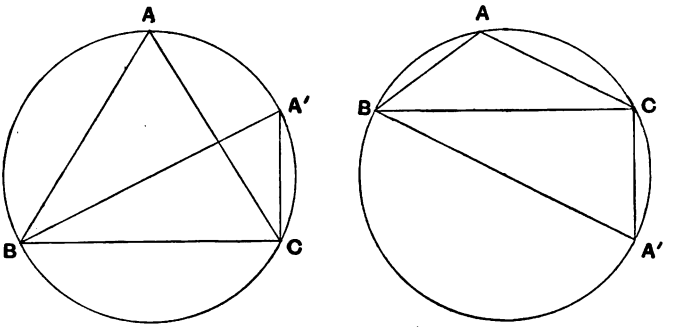
$$\therefore\ r_1 = AY_1 \tan I_1AY_1 = s \tan \tfrac{1}{2}A \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$
$$= BX_1 \cot BI_1X_1 = (s-c)\cot \tfrac{1}{2}B.$$

Or again:

$$r_1 = I_1X_1 = BI_1 \cos BI_1X_1 = \frac{BC \sin BCI_1}{\sin BI_1C} \cdot \cos BI_1X_1$$

$$= \frac{a \cos \tfrac{1}{2}C \cos \tfrac{1}{2}B}{\sin \tfrac{1}{2}(B+C)} = \frac{a \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C}{\cos \tfrac{1}{2}A} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

210. *Expressions for the* Radius (R) *of the* circumscribing circle.



Through B draw the diameter BA' of the circumscribing circle. Join $A'C$.

Then the angle BCA' in a semicircle is a right-angle. And the angle $BA'C$, which is in the same or the opposite segment to BAC is either equal or supplementary to A.

$$\therefore \sin BA'C = \sin A.$$

Now $$\sin BA'C = \frac{BC}{BA'}, \quad \therefore \sin A = \frac{a}{2R}.$$

$$\therefore 2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \ldots\ldots\ldots\ldots (1).$$

$\therefore d$, which was used in Art. 145 for the value of each of these fractions, is the diameter of the circumscribing circle.

Now $$\sin A = \frac{2S}{bc}, \quad \therefore R = \frac{abc}{4S} \ldots\ldots\ldots\ldots (2).$$

The formulæ (1) may be also derived by using the figure of the circumscribing circle in the last chapter.

Cor. The distances of S from the sides of ABC are $R\cos A$, $R\cos B$, $R\cos C$. If A is obtuse, $\cos A$ is negative, and the arithmetic value of the distance of S from BC is $-R\cos A$, S being on the side of BC remote from A.

It will be found convenient to express distances in terms of R and ratios of the angles.

211. *To express r and r_1 in terms of R.*

$$r = \frac{a \sin \frac{1}{2}B \sin \frac{1}{2}C}{\cos \frac{1}{2}A} = \frac{2a}{\sin A} \cdot \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C$$
$$= 4R \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C,$$
$$r_1 = \frac{a \cos \frac{1}{2}B \cos \frac{1}{2}C}{\cos \frac{1}{2}A} = \frac{2a}{\sin A} \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C$$
$$= 4R \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C.$$

212. *The distances of the* orthocentre *from the sides and vertices of the triangle.*

See fig., Art. 191.

$$AO = AM \sec OAM = AB \cos A \operatorname{cosec} C = \frac{c}{\sin C} \cos A = 2R \cos A.$$

If A is obtuse, AO is arithmetically $-2R \cos A$, O being on the side of A remote from BC.

$OL = BL \tan LBO = AB \cos B \cot C = 2R \cos B \cos C$. If B or C is obtuse, OL is arithmetically $-2R \cos B \cos C$.

Cor. The distances of O from the sides are inversely as those of S.

213. *The distances of the* centre of gravity *from the sides and vertices of the triangle.*

See fig., Art. 192.

Since $GD = \frac{1}{3}AD$, $\therefore$ *the distance of G from BC*

$$= \tfrac{1}{3}AL = \tfrac{1}{3}b \sin C = \tfrac{2}{3} \cdot \frac{S}{a} = \tfrac{2}{3}R \sin B \sin C \ldots\ldots(1).$$

Also, producing AD to A', so that $A'D = AD$, $ABA'C$ is a parallelogram, and the angle ABA' is the supplement of A;

$$\therefore 4AD^2 = AA'^2 = b^2 + c^2 + 2bc \cos A \ldots\ldots\ldots\ldots(2),$$
$$= 2b^2 + 2c^2 - a^2 \ldots\ldots\ldots\ldots\ldots\ldots(3).$$
$$\therefore GA = \tfrac{2}{3}AD = \tfrac{1}{3}\sqrt{(2b^2 + 2c^2 - a^2)} \ldots\ldots\ldots\ldots(4).$$

Cor. The distances of G from the sides are inversely as the sides.

214. *Expressions for the* radius (ρ) *of the* Cosine-Circle; *for the distances of its centre from the sides, and for its intercepts on the sides.*

See fig., Art. 193.

Draw KX, KY, KZ perpendiculars on the sides.

Then $\because$ the antiparallels through K, which are diameters of the cosine-circle, cut BC at an inclination A,

$\therefore$ the intercepts on the sides are, respectively,

$$2\rho \cos A,\ 2\rho \cos B,\ 2\rho \cos C,$$

and are $\therefore$ proportional to the cosines of the angles.

[Hence the name of the circle.]

And the perpendiculars on the sides are, respectively,

$$\rho \sin A,\ \rho \sin B,\ \rho \sin C,$$

$$\therefore KX : KY : KZ = \sin A : \sin B : \sin C$$

$$= a : b : c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If now x, y, z be the perpendiculars on the sides from any point within the triangle, it follows, as in Art. 208, that

$$xa + yb + zc = 2 \cdot (\text{area of triangle}) = 2S.$$

$$\therefore \frac{KX}{a} = \frac{KY}{b} = \frac{KZ}{c} = \frac{2S}{a^2 + b^2 + c^2} \quad \ldots\ldots\ldots\ldots (2).$$

This gives the distances of K from the sides.

$$\therefore \rho = \frac{KX}{\sin A} = \frac{a}{\sin A} \cdot \frac{2S}{a^2 + b^2 + c^2} = \frac{4RS}{a^2 + b^2 + c^2} \quad \ldots\ldots\ldots (3)$$

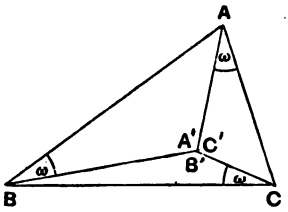
$$= \frac{abc}{a^2 + b^2 + c^2} \quad \ldots\ldots (4).$$

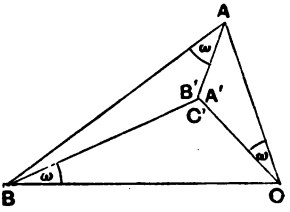
Now $$\frac{a^2 + b^2 + c^2}{4S} = \frac{2bc \cos A + 2ca \cos B + 2ab \cos C}{4S}$$

$$= \cot A + \cot B + \cot C.$$

$$\therefore \frac{\frac{1}{2}a}{KX} = \frac{\frac{1}{2}b}{KY} = \frac{\frac{1}{2}c}{KZ} = \frac{R}{\rho} = \cot A + \cot B + \cot C \quad \ldots\ldots\ldots (5).$$

215. *To find trigonometrically the* Brocard *properties.*





Let $C\Omega$, $A\Omega$, $B\Omega$ subtend equal angles ω at A, B, C respectively. Then clearly the angles $A\Omega B$, $B\Omega C$, $C\Omega A$ must be the supplements of A, B, C.

Thus $$\frac{A\Omega}{\sin\omega} = \frac{c}{\sin A} \text{ and } \frac{A\Omega}{\sin(C-\omega)} = \frac{b}{\sin C},$$

$$\therefore \frac{\sin(C-\omega)}{\sin\omega} = \frac{c}{b}\cdot\frac{\sin C}{\sin A} = \frac{\sin^2 C}{\sin A \sin B} = \frac{\sin C \sin(A+B)}{\sin A \sin B},$$

$$\therefore \cot\omega = \cot A + \cot B + \cot C \quad \ldots\ldots\ldots\ldots(1).$$

If again $B\Omega'$, $C\Omega'$, $A\Omega'$ subtend equal angles ω' at A, B, C respectively, the same equation (1) will give ω'.

$$\therefore \omega = \omega' \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

If ΩX, ΩY, ΩZ be perpendiculars on the sides, since circles go round $AYZ\Omega$, $BZX\Omega$, $CXY\Omega$, it follows that XYZ is similar to CAB, and has Ω and ω for Brocard point and Brocard angle respectively. And since $\Omega X : \Omega C = \sin\omega$, $\therefore$ $\sin\omega$ is the ratio of similitude of XYZ to CAB.

Now $$\Omega X = \Omega C \sin\omega = \frac{b\sin^2\omega}{\sin C}.$$

If $\Omega' X'$ be perpendicular to BC, $\Omega' X' = \dfrac{c\sin^2\omega}{\sin B}$,

$$\therefore \Omega X \,.\, \Omega' X' = \Omega Y \,.\, \Omega' Y' = \Omega Z \,.\, \Omega' Z' = 4R^2 \sin^4\omega.$$

$$\therefore (\tfrac{1}{2}\Omega'\Omega)^2 = (\text{radius of circle } XX'\,YY'\,ZZ')^2 - 4R^2\sin^4\omega$$

$$= R^2\sin^2\omega\,(1 - 4\sin^2\omega) = R^2\sin^2\omega\,(\cos^2\omega - 3\sin^2\omega).$$

$$\therefore \Omega'\Omega = R\sin 2\omega\,\sqrt{(1 - 3\tan^2\omega)} \quad \ldots\ldots(3).$$

216. Since $B\Omega'$ and $C\Omega$ are inclined to BC at equal angles ω, therefore if their point of intersection D' is joined to D (the middle point of BC), we have

$$\frac{\frac{1}{2}a}{DD'} = \frac{\frac{1}{2}b}{EE'} = \frac{\frac{1}{2}c}{FF'} = \cot \omega.$$

Comparing this with Art. 215 (1), and Art. 214 (5), we have

$$KX = DD',\ KY = EE',\ KZ = FF',$$

and

$$\frac{KX}{\frac{1}{2}a} = \frac{KY}{\frac{1}{2}b} = \frac{KZ}{\frac{1}{2}c} = \frac{\rho}{R} = \tan \omega.$$

Thus $\rho = R \tan \omega$.

Cor. (Lemoine-Radius)^2=($\frac{1}{2}$ circumradius)^2+($\frac{1}{2}$ cosine-radius)^2.

$\therefore$ Lemoine-Radius = $\frac{1}{2}R \sec \omega$.

$\therefore$ the sides of $A_1B_1C_1$ and of $A_2B_2C_2$ (where the Lemoine circle cuts ABC) are to those of ABC in the ratio $\frac{1}{2}$ sec ω to 1.

217. *The distance between the Circumcentre and Incentre.*

By Prop. IX. in the last chapter, *the rectangle of the segments of any chord of the circumcircle drawn through the* incentre = $2Rr$.

Draw through I the diameter $DSIE$ of the circumcircle.

Then $$DI \cdot IE + SI^2 = DS^2,$$

$\therefore$ $$SI^2 = DS^2 - DI \cdot IE = R^2 - 2Rr.$$

Similarly $$SI_1^2 = R^2 + 2Rr_1.$$

218. *The distance between the Circumcentre and Orthocentre.*

$$\begin{aligned} SO^2 &= SA^2 + OA^2 - 2SA \cdot OA \cos SAO, \\ &= R^2 + 4R^2 \cos^2 A - 4R^2 \cos A \cos (C - B) \\ &= R^2 - 4R^2 \cos A \{\cos (C + B) + \cos (C - B)\} \\ &= R^2 (1 - 8 \cos A \cos B \cos C). \end{aligned}$$

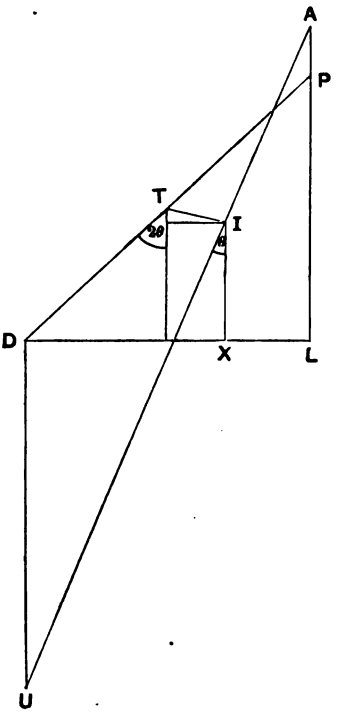
Cor. Since the Nine-Points Centre T bisects SO,

$$\therefore ST^2 = r^2 (1 - 8 \cos A \cos B \cos C),$$

where r = nine-points radius = $\frac{1}{2}R$.

219. *The distance between the nine-points centre and incentre.*

Drop perpendiculars from T on BC, and from I on this perpendicular. Let AI cut the circumcircle in U.



Then, by Prop. IX. of last Chapter, $UI \,.\, IA = 2Rr = 4\tau r$; and by Prop. XI, Cor. II, $\angle\, DPL = 2\,.\, \angle\, UIX = 2\theta$ say.

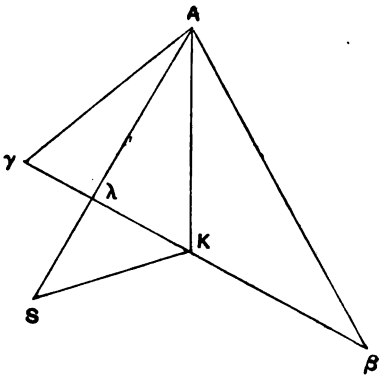
$$\therefore\ TI^2 = (\tau \cos 2\theta - r)^2 + (\tau \sin 2\theta - DX)^2$$
$$= \tau^2 + r^2 - 2\tau r \cos 2\theta - DX\,(2\tau \sin 2\theta - DX)$$
$$= \tau^2 + r^2 - 2\tau r \cos 2\theta - DX \,.\, XL$$
$$= \tau^2 + r^2 - 2\tau r \cos 2\theta - UI \,.\, IA \sin^2\theta$$
$$= \tau^2 + r^2 - 2\tau r;$$

$$\therefore\ TI = \tau - r.$$

Cor. 1. This is another proof that the nine-points circle touches the incircle.

Cor. 2. If we write I_1 for I, r_1 for r, and + for − in the above we have the similar proposition for the escribed circle.

220. *The distance between the Circumcentre and the Cosine-centre.*



Join AK. Through K draw $\beta\gamma$ antiparallel to BC. Join SA cutting $\beta\gamma$ at right-angles in λ.

Then the triangle $A\beta\gamma$ is similar to ABC; so that

$$\frac{A\lambda}{AL} = \frac{AK}{AD} = \frac{\beta\gamma}{BC} = \frac{2\rho}{a}.$$

Thus
$$SK^2 = SA^2 + KA^2 - 2SA \,.\, \lambda A$$
$$= R^2 + \frac{4\rho^2}{a^2} \,.\, AD^2 - 2R \,.\, \frac{2\rho}{a} \,.\, LA$$
$$= R^2 + \frac{\rho^2}{a^2}(2b^2 + 2c^2 - a^2 - 2a^2 - 2b^2 - 2c^2),$$
$$[\text{for } 4AD^2 = 2b^2 + 2c^2 - a^2,$$
$$a \,.\, LA = 2S,$$
$$4R \,.\, S = (a^2 + b^2 + c^2)\,\rho],$$
$$\therefore\ SK^2 = R^2 - 3\rho^2 = R^2(1 - 3\tan^2\omega).$$

Cor. 1. Square on diameter of Brocard Circle $= R^2 - 3\rho^2$.

Cor. 2. Since ΩK and $\Omega' K$ subtend angles equal to ω in the Brocard Circle,

$$\therefore\ \Omega K = \Omega' K = SK \sin\omega = R \sin\omega \sqrt{(1 - 3\tan^2\omega)},$$

$$\text{and } \Omega S = \Omega' S = SK \cos\omega = R\sqrt{(\cos^2\omega - 3\sin^2\omega)},$$

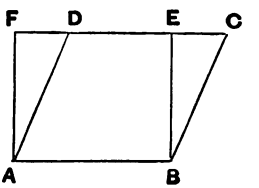
$$\text{and } \Omega\Omega' = SK \sin 2\omega = R \sin 2\omega \sqrt{(1 - 3\tan^2\omega)}.$$

This last result was reached in Art. 215.

§ 2. Areas of Quadrilaterals.

221. *Area of a parallelogram.*

Let $ABCD$ be a parallelogram. Draw AF, BE perpendicular to CD.



Then $ABCD = ABEF = AB \cdot BE = AB \cdot BC \sin BCE$, $\therefore$ *the parallelogram* is measured by *the product of two adjacent sides and the sine of their inclination.*

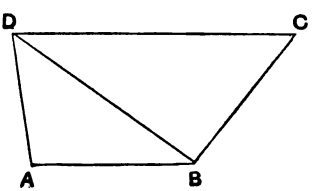
222. *Area of a trapezium* (i.e. a quadrilateral with one pair of opposite sides parallel).

Let $ABCD$ be a quadrilateral in which AB, CD are parallel. Then

$ABCD = \triangle ABD + \triangle BCD$

$= \frac{1}{2}(AB + DC)$ (perpendicular distance between AB and CD),

$= \frac{1}{2}$ (sum of parallel sides) $\times$ altitude.

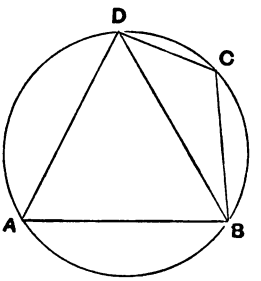


This may be written

$$\tfrac{1}{2}(AB + CD)\, AD \sin ADC \text{ or } \tfrac{1}{2}(AB + CD)\, BC \sin BCD.$$

223. *Area of a quadrilateral inscribable in a circle.*

Let $ABCD$ be a quadrilateral inscribable in a circle; so that *the angles at A and C are supplementary.*



$$\therefore \sin A = \sin C$$

and $$\cos A = -\cos C.$$

Let $AB = a$; $BC = b$; $CD = c$; $DA = d$; and $a + b + c + d = 2s$.

Then $ABCD = \Delta ABD + \Delta BCD$

$= \frac{1}{2} AB \,.\, AD \sin A + \frac{1}{2} BC \,.\, CD \sin C$

$= \frac{1}{2} (ad + bc) \sin A$ ....(1).

Now from ΔABD, $BD^2 = a^2 + d^2 - 2ad \cos A$,

and from ΔBCD, $BD^2 = b^2 + c^2 + 2bc \cos A$;

$$\therefore b^2 + c^2 + 2bc \cos A = a^2 + d^2 - 2ad \cos A,$$

i.e. $$2(ad + bc) \cos A = a^2 + d^2 - b^2 - c^2,$$

and $$2(ad + bc)\, 1 = 2ad + 2bc.$$

$\therefore$ adding and subtracting

$$2(ad + bc)(1 + \cos A) = (a + d)^2 - (b - c)^2,$$
$$2(ad + bc)(1 - \cos A) = (b + c)^2 - (a - d)^2,$$

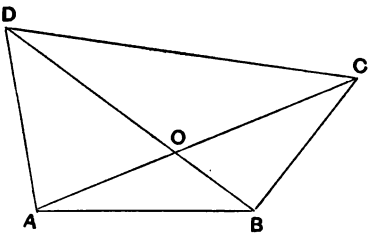
$\therefore$ multiplying

$$4(ad + bc)^2 \sin^2 A$$
$$= (a + d + b - c)(a + d - b + c)(b + c + a - d)(b + c - a + d),$$

$\therefore$ dividing by 16 and taking the square root

Quadrilateral $ABCD = \sqrt{\{(s - a)(s - b)(s - c)(s - d)\}}$.....(2).

224. *The area of any quadrilateral* (in terms of its sides and diagonals).



Let Q = area of $ABCD$.

$AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = e$, $BD = f$.

Let AC, BD cut in O. Then

$$Q = AOB + BOC + COD + DOA$$
$$= \tfrac{1}{2}(AO \,.\, OB \sin AOB + BO \,.\, OC \sin BOC + CO \,.\, OD \sin COD + DO \,.\, OA \sin DOA)$$
$$= \tfrac{1}{2} AC \,.\, BD \sin AOB = \tfrac{1}{2} ef \sin AOB \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now
$$a^2 = AO^2 + BO^2 - 2AO \,.\, BO \cos AOB,$$
$$b^2 = BO^2 + CO^2 - 2BO \,.\, CO \cos BOC,$$
$$c^2 = CO^2 + DO^2 - 2CO \,.\, DO \cos COD,$$
$$d^2 = DO^2 + AO^2 - 2DO \,.\, AO \cos DOA.$$

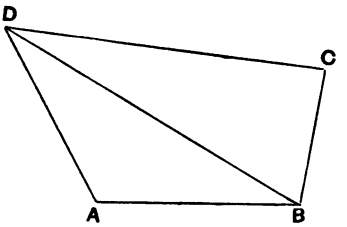
$$\therefore\ b^2 - a^2 + d^2 - c^2 = 2\,(AO \,.\, OB + BO \,.\, OC + CO \,.\, OD + DO \,.\, OA) \cos AOB$$
$$= 2ef \cos AOB.$$
$$\therefore\ 2ef(1 + \cos AOB) = 2ef + b^2 + d^2 - a^2 - c^2,$$
$$2ef(1 - \cos AOB) = 2ef + a^2 + c^2 - b^2 - d^2,$$
$$\therefore\ 16Q^2 = (2ef + b^2 + d^2 - a^2 - c^2)(2ef + a^2 + c^2 - b^2 - d^2) \ldots (2).$$

Cor. 1. Since in a circle $ef = ac + bd$, (Euc. VI. D) therefore his equation reduces to the formula (2) of the preceding article.

Cor. 2. If a circle is inscribable in the Quadrilateral,

$$a + c = b + d,$$
$$\therefore\ Q = \tfrac{1}{2}\sqrt{\{(ef + bd - ac)(ef - bd + ac)\}}.$$

225. *The area of any quadrilateral:* (in terms of its sides and the sum of its opposite angles.)



Let $A + C = 2\theta$, so that $B + D = 2(\pi - \theta)$. Then

$$Q = \triangle ABD + \triangle BCD = \tfrac{1}{2}ad \sin A + \tfrac{1}{2}bc \sin C \ldots\ldots (1).$$

Now $BD^2 = a^2 + d^2 - 2ad \cos A = b^2 + c^2 - 2bc \cos C.$

$$\therefore\ a^2 - b^2 + d^2 - c^2 = 2ad \cos A - 2bc \cos C.$$

$$\therefore\ (a^2 - b^2 + d^2 - c^2)^2 = 4a^2d^2 \cos^2 A - 8abcd \cos A \cos C + 4b^2c^2 \cos^2 C,$$

and $$16Q^2 = 4a^2d^2 \sin^2 A + 8abcd \sin A \sin C + 4b^2c^2 \sin^2 C.$$

$$\therefore\ 16Q^2 + (a^2 - b^2 + d^2 - c^2)^2 = 4a^2d^2 - 8abcd \cos(A + C) + 4b^2c^2.$$

$$\therefore\ 16Q^2 = 4(ad + bc)^2 - (a^2 - b^2 + d^2 - c^2)^2 - 16abcd \cos^2\theta$$

$$= (2ad + 2bc + a^2 - b^2 + d^2 - c^2)(2ad + 2bc - a^2 + b^2 - d^2 + c^2) - 16abcd \cos^2\theta.$$

$$\therefore\ Q^2 = (s - a)(s - b)(s - c)(s - d) - abcd \cos^2\theta \ldots\ldots (2).$$

Cor. 1. If Q is inscribable in a circle, $2\theta = \pi$, $\theta = \frac{1}{2}\pi$ and $\cos\theta = 0$. Hence $Q = \sqrt{\{(s - a)(s - b)(s - c)(s - d)\}}$.

Cor. 2. If a circle is inscribable in Q,

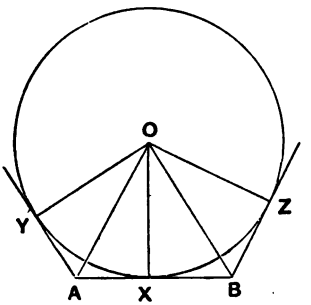
$$s - a = c,\ s - b = d,\ s - c = a,\ s - d = b.$$

$$\therefore\ Q = \sqrt{(abcd)} \,.\, \sin\theta.$$

Cor. 3. A quadrilateral of given sides has its maximum area when it is inscribable in a circle.

§ 3. Regular Polygons.

226. *Formulæ for any polygon circumscribing a circle.*



Let AB be a side of a polygon circumscribing a circle whose centre is O.

Let $s = \frac{1}{2}$ sum of sides of polygon.
r = radius of inscribed circle.
S = area of polygon.

Draw $OX = r$ perpendicular to AB.

Then $\triangle AOB = \frac{1}{2} OX \,.\, AB$.

$\therefore$ area of polygon $= \frac{1}{2}$ (radius × perimeter)

i.e. $S = s \,.\, r$ and $r = \dfrac{S}{s}$ .....(1).

Also $r = OX = OA \sin \dfrac{A}{2} = \dfrac{AB \sin \frac{1}{2}A \sin \frac{1}{2}B}{\sin \frac{1}{2}(A+B)}$ .....(2).

Thus the radius is determined by *one side and the adjacent angles.*

227. *The perimeter and area of a regular polygon in terms of the radius of the* inscribed *circle.*

Take the figure of last article.

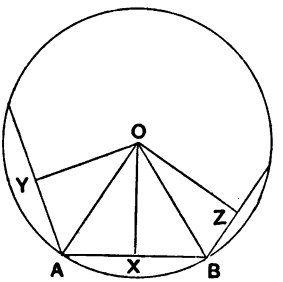
Then the polygon being regular, and containing n sides, (say)

$$n \angle AOB = 4 \text{ right-angles}, \quad \therefore \angle AOB = 2\pi/n.$$

$$\therefore s = \tfrac{1}{2} n \,.\, AB = n \,.\, AX = nr \tan \frac{\pi}{n} \ldots\ldots\ldots\ldots (1).$$

$$S = r \,.\, s = nr^2 \tan \frac{\pi}{n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

228. *The perimeter and area of a regular polygon in terms of the radius of the* circumscribing *circle.*



Let AB be a side of a regular polygon of n sides. Let the centre of the circumscribing circle be O. Draw OX perpendicular to AB.

Then, as in last article, $\angle AOB = 2\pi/n$.

$$\therefore s = \tfrac{1}{2} n \,.\, AB = n \,.\, AX = nR \sin \frac{\pi}{n} \ldots\ldots\ldots\ldots\ldots (1).$$

$$S = n \,.\, \triangle AOB = n \,.\, AX \,.\, OX = nR^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} \ldots\ldots (2).$$

EXAMPLES X.

[The following notation is employed:—area of $ABC=\Delta$; inradius $=r$; circumradius$=R$; cosine-radius$=\rho$; eradii$=r_1, r_2, r_3$; ex-cosine radii$=\rho_1, \rho_2, \rho_3$; half-sum of sides$=s$; and points are lettered as in the last chapter.]

Prove the following statements.

1. $\Delta s = abc \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C.$
2. $\Delta (s-a) = abc \cos \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C.$
3. $\Delta (a+b) = abc \cos \frac{1}{2}C \cos \frac{1}{2}(A-B).$
4. $\Delta (a-b) = abc \sin \frac{1}{2}C \sin \frac{1}{2}(A-B).$
5. $4\Delta = a^2 \sin 2B + b^2 \sin 2A.$
6. $2\Delta \sin (A-B) = (a^2-b^2) \sin A \sin B.$
7. $16\Delta^2 = (a^2+b^2+c^2)^2 - 2(a^4+b^4+c^4).$
8. $abc = 4R\Delta = 4srR = \rho (a^2+b^2+c^2).$
9. $\Delta = Rr (\sin A + \sin B + \sin C).$
10. $r^2 = \Delta \tan \frac{1}{2}A \tan \frac{1}{2}B \tan \frac{1}{2}C.$
11. $s = 4R \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C.$
12. $abc\, r = 4R (s-a)(s-b)(s-c).$
13. $2r + 2R = a \cot A + b \cot B + c \cot C.$
14. $4R \sin A \sin B \sin C = a \cos A + b \cos B + c \cos C.$
15. $\Delta = 2R^2 \sin A \sin B \sin C.$
16. $2\Delta = \rho (a \sin A + b \sin B + c \sin C).$
17. $\dfrac{\rho}{R} = \dfrac{\sin A \sin B \sin C}{1 + \cos A \cos B \cos C} = \tan \omega.$
18. $\cot \omega = \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C + \cot A \cot B \cot C.$
19. $\operatorname{cosec}^2 \omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C.$
20. $\dfrac{1}{\rho} + \dfrac{2}{a} + \dfrac{2}{b} + \dfrac{2}{c} = \dfrac{\Delta}{Rr^2} = \dfrac{s^2}{R\Delta}.$
21. $\dfrac{1}{\rho^2} + \dfrac{1}{R^2} = \dfrac{4}{a^2} + \dfrac{4}{b^2} + \dfrac{4}{c^2}.$

22. $\dfrac{1}{\rho^2} - \dfrac{1}{R^2} = \dfrac{2a^2}{b^2c^2} + \dfrac{2b^2}{c^2a^2} + \dfrac{2c^2}{a^2b^2}$.

23. $\dfrac{a}{bc} + \dfrac{b}{ca} + \dfrac{c}{ab} = \dfrac{1}{\rho}$.

24. $rr_1r_2r_3 = \Delta^2$.

25. $r_2r_3 + r_3r_1 + r_1r_2 = s^2$.

26. $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{1}{r}$.

27. $r_1 + r_2 + r_3 - r = 4R$.

28. $\dfrac{r_1}{bc} + \dfrac{r_2}{ca} + \dfrac{r_3}{ab} = \dfrac{1}{r} - \dfrac{1}{2R}$.

29. $\dfrac{rr_1}{r_2r_3} = \tan^2\dfrac{A}{2}$.

30. $\rho_1 = \dfrac{abc}{b^2 + c^2 - a^2} = R\tan A$.

31. $\dfrac{1}{\rho_1} + \dfrac{1}{\rho_2} + \dfrac{1}{\rho_3} = \dfrac{1}{\rho}$.

32. $\dfrac{1}{\rho_2\rho_3} + \dfrac{1}{\rho_3\rho_1} + \dfrac{1}{\rho_1\rho_2} = \dfrac{1}{R^2}$.

33. $\rho_2\rho_3 + \rho_3\rho_1 + \rho_1\rho_2$
$= 3R^2 + R^2(\cos^2 A + \cos^2 B + \cos^2 C)\sec A \sec B \sec C$.

34. $\dfrac{1}{r_2r_3} + \dfrac{1}{r_3r_1} + \dfrac{1}{r_1r_2} + \dfrac{1}{r^2} = \dfrac{bc + ca + ab}{\Delta^2}$.

35. $4r(r_1 + r_2 + r_3) = 2bc + 2ca + 2ab - a^2 - b^2 - c^2$.

36. $\dfrac{ar_2r_3}{r_2 + r_3} = \dfrac{br_3r_1}{r_3 + r_1} = \dfrac{cr_1r_2}{r_1 + r_2} = \dfrac{r_1r_2r_3}{\sqrt{(r_2r_3 + r_3r_1 + r_1r_2)}}$.

37. $\sqrt{(ar_1)} + \sqrt{(br_2)} + \sqrt{(cr_3)} - \sqrt{(abc/r)}$
$= 8\sqrt{(Rs)}\sin(45° - \tfrac{1}{4}A)\sin(45° - \tfrac{1}{4}B)\sin(45° - \tfrac{1}{4}C)$.

38. $2r\rho(r_1 + r_2 + r_3) = 2\rho s^2 - abc$.

39. $\Delta^2\left(\dfrac{1}{r_2} - \dfrac{1}{r_3}\right)\left(\dfrac{1}{r_3} - \dfrac{1}{r_1}\right)\left(\dfrac{1}{r_1} - \dfrac{1}{r_2}\right)$
$= a^2\left(\dfrac{1}{r_3} - \dfrac{1}{r_2}\right) + b^2\left(\dfrac{1}{r_1} - \dfrac{1}{r_3}\right) + c^2\left(\dfrac{1}{r_2} - \dfrac{1}{r_1}\right)$
$= 2aR\left(\dfrac{1}{\rho_2} - \dfrac{1}{\rho_3}\right) + 2bR\left(\dfrac{1}{\rho_3} - \dfrac{1}{\rho_1}\right) + 2cR\left(\dfrac{1}{\rho_1} - \dfrac{1}{\rho_2}\right)$.

40. $r_1 \cot \frac{1}{2}A = r_2 \cot \frac{1}{2}B = r_3 \cot \frac{1}{2}C = r \cot \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C.$

41. $\rho_1 \cot A = \rho_2 \cot B = \rho_3 \cot C = \rho \cot \omega.$

42. $\rho_2 + \rho_3 = \frac{1}{2}a \sec B \sec C.$

43. $r_2 + r_3 = a \cot \frac{1}{2}A.$

44. $\rho_1 + \rho_2 + \rho_3 = \dfrac{\rho}{1 - 2R\rho/\Delta}.$

45. $$\frac{r}{\rho} = \frac{2\cos A}{\cot \frac{1}{2}B + \cot \frac{1}{2}C} + \frac{2\cos B}{\cot \frac{1}{2}C + \cot \frac{1}{2}A} + \frac{2\cos C}{\cot \frac{1}{2}A + \cot \frac{1}{2}B} = \frac{a \sin A + b \sin B + c \sin C}{a + b + c}.$$

46. In no triangle is r greater than $\dfrac{R}{2}$, nor ρ greater than $\dfrac{R}{\sqrt{3}}$.

47. Given the inradius r, the circumradius R, and the area Δ of a triangle, show that its sides are the roots of the equation

$$x^3 - 2x^2\Delta/r + x(r^2 + 4rR + \Delta^2/r^2) = 4\Delta R.$$

48. Given the half-sum of sides s, the half-sum of squares on sides σ^2, and the area Δ of a triangle, show that the radii of its escribed circles are the roots of the equation

$$x^3/s + xs = x^2(s^2 - \sigma^2)/\Delta + \Delta.$$

49. If the sides of a triangle are the roots of the equation $x^3 + px = qx^2 + v$, its cosine-radius is $v/(q^2 - 2p)$ and the rectangle contained by its inradius and circumradius is $v/2q$.

50. If the squares on the sides of a triangle are the roots of the equation $x^3 + px = qx^2 + v^2$, its cosine-radius is v/q and its circumradius is $\dfrac{v}{\sqrt{(4p - q^2)}}$.

51. In an equilateral triangle, the incircle coincides with the nine-points circle, the cosine-circle with the Lemoine circle, and the centres of the escribed circles with those of the ex-cosine circles. Also $R = \dfrac{a}{\sqrt{3}}$, $\rho = \dfrac{a}{3}$, $r = \dfrac{a}{2\sqrt{3}}$, $r_1 = \dfrac{\sqrt{3}}{2}a$, $\rho_1 = a$, $\omega = 30°$.

52. In a right-angled triangle, the diameter of the circumcircle is the hypothenuse and that of the cosine-circle is the perpendicular from the right-angle upon the hypothenuse. Also, if C is the right-angle,

$$2r + 2R = 2r_3 - 2R = a + b,$$

$$r_1 + r_2 = r_3 - r = c,$$

$$\rho_1\rho_2 = R^2, \qquad \rho_3 = \infty,$$

$$2\tan\omega = \sin 2A.$$

53. In the ambiguous case, given b, c, B; show that the circumradii of the two triangles are equal; the ratio of their areas is $1 : m$ where

$$4c^2m\cos^2 B = (c^2 - b^2)(1+m)^2,$$

and the distance between their orthocentres is

$$2\sqrt{(b^2\cot^2 B - c^2\cos^2 B)}.$$

54. $\dfrac{\cos A}{AL} + \dfrac{\cos B}{BM} + \dfrac{\cos C}{CN} = \dfrac{1}{R}.$

55. $\dfrac{1}{AL} + \dfrac{1}{BM} + \dfrac{1}{CN} = \dfrac{1}{r}.$

56. $\dfrac{MN^2}{a^2} + \dfrac{NL^2}{b^2} + \dfrac{LM^2}{c^2} + 2\dfrac{MN \cdot NL \cdot LM}{abc} = 1.$

57. $a \cdot IA^2 + b \cdot IB^2 + c \cdot IC^2 = abc.$

58. $II_1 : II_2 : II_3 = \sin\tfrac{1}{2}A : \sin\tfrac{1}{2}B : \sin\tfrac{1}{2}C.$

59. $II_1 \cdot II_2 \cdot II_3 = 16R^2r.$

60. If AI, BI, CI cut the sides in H_1, H_2, H_3,

$$\frac{\cos\frac{1}{2}A}{AH_1} + \frac{\cos\frac{1}{2}B}{BH_2} + \frac{\cos\frac{1}{2}C}{CH_3} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

61. $SO^2 = \tfrac{1}{4}(5 + 9\cot^2 A)\, a^2 - (b^2 + c^2).$

62. $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2.$

63. $AX^2 = (s-a)^2 + \dfrac{r}{R}bc.$

64. $IA \cdot IB \cdot IC = abc\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B\tan\tfrac{1}{2}C.$

65. $IO^2 = 2r^2 - 4R^2 \cos A \cos B \cos C$; hence, from the known values of SO^2 and SI^2, find IT.

66. Of the triangle LMN, the side opposite A is $a \cos A$, the half sum of sides is $2R \sin A \sin B \sin C$, the area is $2\Delta \cos A \cos B \cos C$, the inradius is $2R \cos A \cos B \cos C$, the eradius is $2R \cos A \sin B \sin C$.

67. Of the triangle $I_1I_2I_3$, the angle is $90^0 - \frac{1}{2}A$, the side is $a \operatorname{cosec} \frac{1}{2}A$, the area is $2Rs$, the circumradius is $2R$, the inradius is

$$\frac{4R \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C}{\cos \frac{1}{2}A + \cos \frac{1}{2}B + \cos \frac{1}{2}C}.$$

68. Of the triangle XYZ, the side is $2r \cos \frac{1}{2}A$, the area is $\Delta \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$.

69. Of the triangle $K_1K_2K_3$, the angle is $180^0 - 2A$, the side is $\frac{1}{2}a \sec B \sec C$, the half-sum of sides is $R \tan A \tan B \tan C$, the area is $\frac{1}{2}\Delta \sec A \sec B \sec C$, the circumradius is $\frac{1}{4}R \sec A \sec B \sec C$.

70. If AI, BI, CI cut the opposite sides in $H_1H_2H_3$,

$$\Delta H_1H_2H_3 : \Delta ABC = 2abc : (b+c)(c+a)(a+b)$$
$$H_2H_3 : SI_1 = 4\Delta ABC : (c+a)(a+b).$$

71. If AK, BK, CK cut the opposite sides in D', E', F'',

(1) $\Delta D'E'F'' : \Delta ABC = 2a^2b^2c^2 : (b^2+c^2)(c^2+a^2)(a^2+b^2)$,

(2) $AD' : AD = 2bc : b^2+c^2$,

(3) $AK : AD = 2bc : a^2+b^2+c^2$,

(4) $$\frac{AD}{AD' \sin A} + \frac{BE}{BE' \sin B} + \frac{CF}{CF'' \sin C} = \frac{2R}{\rho} = 2 \cot \omega,$$

(5) $$\frac{AK \sin A}{AD} = \frac{BK \sin B}{BE} = \frac{CK \sin C}{CF} = \frac{\rho}{R} = \tan \omega.$$

72. If A, B, C are the angles of any triangle, find the angles of the triangle whose sides are proportional to

(1) $\cos \frac{1}{2}A$, $\cos \frac{1}{2}B$, $\cos \frac{1}{2}C$. (2) $\cos \frac{1}{2}A$, $\sin \frac{1}{2}B$, $\sin \frac{1}{2}C$.

(3) $\sin 2A$, $\sin 2B$, $\sin 2C$.

73. If s', r', R' be the lengths in the Pedal Triangle corresponding to s, r, R in the primitive triangle, then

$$\tan\omega = \frac{s'}{4R' + r'}.$$

74. The radius of that Tucker's circle whose centre is midway between the Brocard points is $R\sin\omega$.

75. If the sides of a triangle are in H.P., so also are the areas of its escribed circles.

76. If AL, BM, CN meet the circumcircle in A', B', C' then

$$\frac{\cos(B-C)}{AA'} = \frac{\cos(C-A)}{BB'} = \frac{\cos(A-B)}{CC'} = \frac{2ABC}{abc}.$$

77. If tangents to the incircle be drawn so as to cut off from the corners triangles similar to ABC, the circumradii of these triangles are inversely as the eradii of ABC.

78. If 2ϕ be the angle at which the circumcircle cuts the ecircle, $\cos\phi = \frac{1}{2}\sqrt{\frac{r_1}{R}}$.

79. The area of the triangle formed by the points of contact of either of the four circles which touch the three sides of a triangle is to the area of the given triangle as the circumradius of the former is to the circumdiameter of the latter.

80. If a, b, c are in A.P., then $2r$, IB, R are in G.P.

81. If SI cuts the incircle in U and V, then the product of the rectangles of the segments of any chords of the circumcircle through U and $V = r^3(r + 4R)$.

82. The square on the inradius of the triangle whose sides are $b + c$, $c + a$, $a + b$ is equal to $2Rr$.

83. The perpendicular from G on BC cuts it in the ratio

$$3a^2 + c^2 - b^2 \ : \ 3a^2 + b^2 - c^2.$$

84. The perpendicular from K on BC cuts it in the ratio

$$a^2 + 3c^2 - b^2 \ : \ a^2 + 3b^2 - c^2.$$

85. Of the triangle formed by the feet of the perpendiculars from G on the sides, the circumradius is $\frac{2}{3} \cdot \frac{def}{\sigma^2}$ and the area is $\frac{\Delta^2}{9R\rho}$, where d, e, f are the distances of the vertices from the middle points of the opposite sides, and $2\sigma^2 = a^2 + b^2 + c^2$.

86. Of the triangle formed by the feet of the perpendiculars from K on the sides, the circumradius is $\frac{2}{3} \cdot \frac{def}{\sigma^2}$, and the area is $\frac{3\Delta^3}{\sigma^4}$.

87. Of the triangle formed by the feet of the perpendiculars from K_1 on the sides the area is $\frac{3\Delta^3}{(\sigma^2 - a^2)^2}$; and the intercepts on the sides by the ex-cosine circle (centre K_1) are to one another as $\cos A$, $\cos B$, $\cos C$.

88. If AS, BS, CS cut the sides in L', M', N', then

(1) $\frac{2R}{AL'} = 1 + \cot B \cot C$, (2) $\frac{1}{AL'} + \frac{1}{BM'} + \frac{1}{CN'} = \frac{2}{R}$,

(3) $\Delta L'M'N' : \Delta ABC$
$= 2 \cos A \cos B \cos C : \cos(A - B) \cos(A - C) \cos(B - C)$.

89. Show that

(1) DK passes through the middle point λ of AL.

(2) $K\lambda : DK = bc \cos A : a^2$.

(3) SK cuts LA in the ratio $\cos(180^0 - 2A) : 1$.

(4) SK passes through the orthocentre of LMN.

(5) If O' is the orthocentre of LMN, $SO' : SK = \Delta : R\rho$.

90. The triangles formed by the feet of the perpendiculars from the two Brocard points are each similar to ABC, their sides being reduced in the ratio of the sine of the Brocard angle.

91. The tangents from A, B, C to the nine-points circle are as $\sqrt{(\cot A)}$, $\sqrt{(\cot B)}$, $\sqrt{(\cot C)}$; and those to the cosine circle are as $\operatorname{cosec} A \sqrt{(\cot A)}$, $\operatorname{cosec} B \sqrt{(\cot B)}$, $\operatorname{cosec} C \sqrt{(\cot C)}$.

92. If $A\Omega$ cut the Brocard Circle in E', $A\Omega \cdot AE' = \frac{bc\rho}{a}$.

Hence the tangents from A, B, C to the Brocard circle are inversely as a, b, c.

93. ω is never greater than 30°.

94. The circumradii of the triangles into which either Brocard point divides the primitive triangle are each equal to a half the product of a side into the cosecant of an adjacent angle. Hence the product of the three circumradii $= R^3$.

95. The area of the triangle which contains the angle ω at the vertex A is $\Delta \sin^2 \omega \operatorname{cosec}^2 A$.

96. $\cot ABK + \cot BCK + \cot CAK$
$= \cot ACK + \cot CBK + \cot BAK = 3 \cot \omega$.

97. AK cuts BC in the ratio $c^2 : b^2$. Hence

(1) The Lemoine Circle cuts BC in the ratio $c^2 : a^2 : b^2$.

(2) The intercepts on the three sides by the Lemoine Circle are as $a^3 : b^3 : c^3$.

[From this last property the Lemoine Circle is called the Triplicate Ratio circle.]

98. The cosine-circle cuts BC in the ratio $\cot B : \cot A : \cot C$.

99. The perimeter and area of the hexagon in which the cosine circle cuts the sides are $4(R + r) \tan \omega$, and $2\Delta \tan^2 \omega$ respectively.

100. If $\beta\gamma'$, $\gamma\alpha'$, $\alpha\beta'$ be the antiparallels to the sides through K, the square on the cosine-diameter $= B\alpha' \cdot C\alpha = C\beta' \cdot A\beta = A\gamma' \cdot B\gamma$.

101. Of the Lemoine Hexagon, the perimeter is $\frac{a^3 + b^3 + c^3 + 3abc}{a^2 + b^2 + c^2}$; the area is $\frac{\Delta (a^4 + b^4 + c^4 + a^2b^2 + b^2c^2 + c^2a^2)}{(a^2 + b^2 + c^2)^2}$;

the sides are as

$$\sin (A - \omega) : \sin \omega : \sin (B - \omega) : \sin \omega : \sin (C - \omega) : \sin \omega.$$

102. The inradii of $X_1Y_1Z_1$, $X_2Y_2Z_2$, $X_3Y_3Z_3$, are to one another inversely as $1 - \tan \frac{1}{4}A$, $1 - \tan \frac{1}{4}B$, $1 - \tan \frac{1}{4}C$.

103. Through A, B, C the lines A_1B_1, B_1C_1, C_1A_1 are drawn perpendicular respectively to the sides AB, BC, CA of ABC: show that the circumradius of $A_1B_1C_1$ is to that of ABC as

$$\sin^2 A + \sin^2 B + \sin^2 C \text{ is to } 2 \sin A \sin B \sin C.$$

104. If the A-escribed circle is equal to the circumcircle, $\cos A = \cos B + \cos C$.

105. As any point P moves on the circumcircle, the quantity $AP^2 . \Delta BSC + BP^2 . \Delta CSA + CP^2 . \Delta ASB$ remains constant.

106. If from the primitive triangle a tangent to the incircle parallel to BC cut off a new triangle, and from the triangle so formed another triangle is similarly cut off, show that the aggregate areas of all the triangles is $\dfrac{\Delta s^2}{a(b+c)}$.

107. If $IS = IO$, one of the angles of the triangle is 60°.

108. The distance between two points is a, their distances from a given straight line are b and c. Of all the triangles that can be described having the same base a and vertex lying in the given straight line, the area of that which has the greatest vertical angle is $\frac{1}{2}a\sqrt{(bc)}$.

109. If S_1 be taken on AS, so that SS_1 is cut by BC in the same ratio as it is cut at A, then

(1) $AS_1 \equiv R_1$ (say) $= R \tan B \tan C$.

(2) $SS_1 = R \cos A \sec B \sec C$.

(3) S_2S_3, S_3S_1, S_1S_2 pass through A, B, C respectively.

(4) The perpendiculars from S_1 upon the sides are as those from S.

(5) The intercepts on the sides by the circle with centre S_1 and radius R_1 are to one another as the sides.

(6) The tangents to this circle where it cuts the sides are antiparallels to the sides.

(7) $\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} = \dfrac{1}{R}$.

110. If x, y, z are the perpendiculars from any point upon the sides of ABC, then

(1) If the point is *within* the triangle,

$$ax + by + cz = 2\Delta.$$

(2) If the point is within the space formed by producing AB and AC,

$$by + cz - ax = 2\Delta.$$

(3) If the point is within the space formed by producing BA and CA,

$$ax - by - cz = 2\Delta.$$

111. Show that four points may be found so that $x : y : z$ is in any given ratio: and that the line joining any two out of the four points passes through a vertex of the triangle and is cut at the vertex in the same ratio as by the opposite side.

112. If J, J_1, J_2, J_3 be the four points in the last example, viz. such that AJJ_1, J_2AJ_3, &c., are straight lines, and if through J lines $B'C''$, $C'A''$, $A'B''$ be drawn parallel to BC, CA, AB respectively and cutting the sides in the hexagon $A'B'C'A''B''C''$, then $A'A''$, $B'B''$, $C'C''$ will be parallel to J_2J_3, J_3J_1, J_1J_2 respectively.

113. In the last example, show that four circles with centres J, J_1, J_2, J_3 and centres of similitude A, B, C may be described all of which will cut the sides at the same angles; and that their radii ρ, ρ_1, ρ_2, ρ_3 will be such that

$$\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3}.$$

[Particular cases of propositions 111, 112, 113 are given by $II_1I_2I_3$, $KK_1K_2K_3$, and $SS_1S_2S_3$. (Ex. 109.)]

114. If J be any point, ABC any triangle, and if $a \cdot AJ^2 \equiv X$, $b \cdot BJ^2 \equiv Y$, $c \cdot CJ^2 \equiv Z$, and $abc \equiv T$, then

$$X^2 + Y^2 + Z^2 + T^2$$
$$= 2\cos A\,(YZ + TX) + 2\cos B\,(ZX + TY) + 2\cos C\,(XY + TZ).$$

Hence find the radius of the circle circumscribing ABC.

115. The distance from A of the point at which the sides of ABC subtend angles α, β, γ is

$$\frac{2\Delta \sin\alpha + \frac{1}{2}(a^2 - b^2 - c^2)\cos\alpha}{\sqrt{\{\sin\alpha \sin\beta \sin\gamma\,(4\Delta - a^2\cot\alpha - b^2\cot\beta - c^2\cot\gamma)\}}}.$$

116. If x, y, z be perpendiculars from any point upon the sides of a triangle, $(x^2+y^2+z^2)(a^2+b^2+c^2)$ is never less than $4\Delta^2$, and has its minimum value at K.

117. If through any point P within a triangle AP, BP, CP cut the sides in A', B', C', then

$$BC' \,.\, CA' \,.\, AB' = CB' \,.\, BA' \,.\, AC'.$$

Also if $BA' : CA' = n : m$ and $CB' : AB' = l : n$, then

$$\Delta A'B'C' : \Delta ABC = 2lmn : (m+n)(n+l)(l+m).$$

And $AP^2 = \dfrac{2\Delta mn}{(l+m+n)\,l} \cdot \dfrac{\sin\alpha}{\sin\beta \sin\gamma}$, where α, β, γ are the angles subtended by the sides at P.

118. If the perimeter of the triangle formed by the feet of the perpendiculars from any point P on the sides is p, then the least value of $PA^2 + PB^2 + PC^2$ is

$$\frac{p^2}{\sin^2 A + \sin^2 B + \sin^2 C},$$

and for this least value $PA : PB : PC = a : b : c$.

119. The area of the triangle formed by the feet of the perpendiculars from any point P within ABC is less than $\frac{1}{2}$ the area of ABC by

$$\tfrac{1}{8}(AP^2 \,.\, \sin 2A + BP^2 \,.\, \sin 2B + CP^2 \,.\, \sin 2C).$$

120. Lines $B'C'$, $C'A'$, $A'B'$ are drawn parallel to the sides BC, CA, AB at distances x, y, z respectively; find the area of $A'B'C'$.

If eight triangles be so formed the mean of their perimeters is equal to the perimeter of ABC, but the mean of their areas exceeds its area Δ by

$$\frac{a^2x^2 + b^2y^2 + c^2z^2}{4\Delta}.$$

[In the following examples (121—124), $ABCD$ is a quadrilateral inscribable in a circle.]

121. If BA, CD be produced to meet in O, and if $AO = x$, $DO = y$, show that x and y are given by the equations

$$x = \frac{d}{b}(c + y); \quad y = \frac{d}{b}(a + x).$$

Hence find the area of $ABCD$ from the formulæ giving the areas of OAD and OBC.

122. The circumradius of $ABCD$ is

$$\frac{1}{4}\sqrt{\left\{\frac{(ab + cd)(ac + db)(ad + bc)}{(s-a)(s-b)(s-c)(s-d)}\right\}}$$

and
$$BD = \sqrt{\left\{\frac{(ab + cd)(ac + bd)}{bc + ad}\right\}}.$$

123. If A, B, C, D be the areas of the triangles whose bases are a, b, c, d and vertex the intersection of the diagonals, then

$$\frac{A}{a^2 . bd} = \frac{B}{b^2 . ca} = \frac{C}{c^2 . db} = \frac{D}{d^2 . ac} = \frac{Q}{(bc + ad)(ab + cd)}.$$

124. If BA, CD intersect in E; BC, AD in F; and AC, BD in G, then

$$\text{area } EFG : \text{area } ABCD = 2abcd : (d^2 \sim b^2)(c^2 \sim a^2).$$

125. Find the ratio of the perimeter of a regular polygon to the diameter of its circumcircle when the polygon has 6, 8, 10, 12, 20 sides. Evaluate the surd expressions in each case to four decimal places.

126. Show that the square described about a circle is $\frac{4}{3}$ of the dodecagon inscribed in the circle.

127. If R and r be the radii of the inscribed and circumscribed circles of a polygon of n sides, each $= a$,

$$R + r = \frac{a}{2}\cot\frac{90^\circ}{n}.$$

128. The area of an irregular polygon of an even number of sides described about a circle is equal to the radius × the sum of every alternate side.

129. If a, b, c, d be the sides of a quadrilateral circumscribing a circle, and if β, γ be the angles contained by a, b and by c, d respectively,

$$ab \sin^2 \tfrac{1}{2}\beta = cd \sin^2 \tfrac{1}{2}\gamma.$$

Hence show that the area of the quadrilateral is

$$\sqrt{(abcd)} \,.\, \sin \tfrac{1}{2}(\beta + \gamma).$$

CHAPTER XI.

INDICES OR LOGARITHMS; AND MATHEMATICAL TABLES.

Negative Symbols.

If $a>c$, the symbol $a-c$ has an intelligible meaning. In this case we may easily show that

I. The *addition* of $a-c$ (to any number whatever) is equivalent to the *addition* of a followed by the *subtraction* of c. Or, in symbols,

$$+(a-c)=+a-c \quad \text{.....I.}$$

II. The *subtraction* of $a-c$ (from any number $>a$) is equivalent to the *subtraction* of a followed by the *addition* of c. Or, in symbols,

$$-(a-c)=-a+c \quad \text{.....II.}$$

I. and II. here represent equivalences of *operation*.

But if $a<c$, the symbol $a-c$ has no intelligible meaning by itself. It is convenient, however, to be able to use the above fundamental equivalences in *all* cases whatever.

Putting, then, $a=0$; $(a-c)$ becomes $(0-c)$. Such a quantity is called a *negative quantity*. It may be written for brevity $-c$, or still better $\bar{c}$. The sign $-$ is here called a *sign of affection*. It is useful to write it *over* a number to distinguish it from the sign $-$ used to denote the operation of subtraction.

Putting $a=0$ on the *right*-hand side of I. and II. (since the addition or subtraction of 0 has no effect), the right hand of (I.) becomes $-c$; that of II. becomes $+c$. Hence (I.) and (II.) become

$$+\bar{c}=-c \text{ and } -\bar{c}=+c.$$

In words:

The addition or subtraction of a negative symbol is interpreted to mean the subtraction or addition of the corresponding positive.

It remains to interpret *multiplication* involving a negative quantity.

If $a>b$ and $c>d$, we may prove that

$$(a-b)(c-d)=ac-bc-ad+bd \quad \text{.....III.}$$

It is convenient to use this equation for all cases. Hence if $a<b$ or $c<d$ we so interpret multiplication of negatives that this shall always hold. Thus

Put $b=0$ and $d=0$; then $ac=ac-0-0+0=ac$.
Put $a=0$ and $d=0$; then $\bar{b}c=0-bc-0+0=\bar{bc}$.
Put $b=0$ and $c=0$; then $a\bar{d}=0-0-ad+0=\bar{ad}$.
Put $a=0$ and $c=0$; then $\bar{b}\bar{d}=0-0-0+bd=bd$.

Thus the multiplication of a negative by a positive is *negative;* and the multiplication of a negative by a negative is *positive.*

§ 1. Index Notation.

229. If m is any positive integer, the symbol x^m is used to denote $x \times x \times x \times \ldots\ldots$ to m factors.

Here x is called the *Base*; m, the *Index*; and x^m, the m^{th} *Power* of x.

An m^{th} *Root* of a given quantity *means a quantity whose* m^{th} *Power is equal to the given quantity.*

The symbol $\sqrt[m]{x}$ denotes an m^{th} Root of x; i.e. $(\sqrt[m]{x})^m = x$.

230. *To determine how many positive or negative roots a given positive or negative quantity has.*

The rules for multiplying positive or negative quantities are

I. The product of *positive* quantities is *positive.*

II. A change in sign of *one* factor changes the sign of the *product.*

Hence a product containing an **even** number of negative factors is **positive**; and a product containing an **odd** number of negative factors is **negative**.

From this it follows conversely that

(1) If m is *odd*, and x *positive*, there is *no negative* m^{th} root of x.

(2) If m is *odd*, and x *negative*, there is *no positive* m^{th} root of x.

(3) If m is *even*, and x *negative*, there is *no positive* and there is *no negative* m^{th} root of x.

(4) There cannot be two *different* m^{th} roots of x having the same sign. For, if possible, let y and z be two such roots, so that $y^m = z^m = x$: then

$$y^m - z^m \equiv (y-z)(y^{m-1} + zy^{m-2} + z^2y^{m-3} + \ldots\ldots + z^{m-1}) = 0.$$

But, by the rules of signs, every term in the second factor has the same sign, $\therefore$ this second factor cannot be zero. $\therefore y - z = 0$, i.e. $y = z$. Hence the two roots supposed to be different are not different.

Thus, considering positive or negative values of $\sqrt[m]{x}$, we have the following results*:—

If m is odd, and x positive, $\sqrt[m]{x}$ can have only *one* value, and this *positive*.

If m is odd, and x negative, $\sqrt[m]{x}$ can have only *one* value, and this *negative*.

If m is even, and x positive, $\sqrt[m]{x}$ can have only *two* values, one positive and the other negative.

If m is even, and x negative, $\sqrt[m]{x}$ can have *no* values, positive or negative.

231. The symbol $\sqrt[m]{x}$ may be used for the present to denote the *positive* value of the m^{th} root of x, if x is positive; and the *negative* value of the m^{th} root of x, if x is negative (and m odd).

* Whether there are roots which are neither positive nor negative, is not here discussed.

232. The method of finding the arithmetical value of $\sqrt[m]{x}$, when m and x have any particular values, is not explained here.

But it is important to observe that its value cannot be *exactly*, but only *approximately*, determined in most cases.

A number which cannot be arithmetically expressed by a fraction having a finite integral numerator and denominator is called an **Irrational** or **Incommensurable*** number.

Thus, $\sqrt[3]{2197}=13$ and is, therefore, rational. But $\sqrt[3]{2}=1{\cdot}414$ &c., cannot be exactly evaluated, and is therefore irrational.

Laws involving the same *indices but* different *bases.*

233. The power [or root] of a product [or quotient] of two quantities is equal to the product [or quotient] of the corresponding powers [or roots] of the two quantities. Thus

$$(1)\quad (x\times y)^m = x^m \times y^m;$$
$$(2)\quad (x\div y)^m = x^m \div y^m;$$
$$(3)\quad \sqrt[m]{(x\times y)} = \sqrt[m]{x}\times\sqrt[m]{y};$$
$$(4)\quad \sqrt[m]{(x\div y)} = \sqrt[m]{x}\div\sqrt[m]{y}.$$

For $(x\times y)^m$ means $(x\times y)\times(x\times y)\times \ldots$ to m factors

$= (x\times x\times \ldots$ to m factors$)\times(y\times y\times \ldots$ to m factors$)$

$= x^m \times y^m$ .....(1).

Similarly $(x\div y)^m = x^m \div y^m$ .....(2).

In (1) write $\sqrt[m]{x}$ and $\sqrt[m]{y}$ instead of x and y respectively.

Thus $(\sqrt[m]{x}\times\sqrt[m]{y})^m = (\sqrt[m]{x})^m \times (\sqrt[m]{y})^m = x\times y.$

$\therefore$ taking the m^{th} root of both sides

$$\sqrt[m]{x}\times\sqrt[m]{y} = \sqrt[m]{(x\times y)} \quad \ldots\ldots(3).$$

Similarly $$\sqrt[m]{x}\div\sqrt[m]{y} = \sqrt[m]{(x\div y)} \quad \ldots\ldots(4).$$

* That is, incommensurable with *unity* as explained in Art. 31.

Laws involving different *indices but the* same *bases.*

Law I. $x^m \times x^n = x^{m+n}$.

Law II. If m is greater than n, $x^m \div x^n = x^{m-n}$.

Law III. $(x^m)^n = x^{m \times n}$.

Law IV. If m is divisible by n, $\sqrt[n]{(x^m)} = x^{m \div n}$.

234. Law I. $x^m \times x^n = x^{m+n}$.

For x^m means $x \times x \times \ldots$ to m factors;

x^n means $x \times x \times \ldots$ to n factors.

$\therefore x^m \times x^n = (x \times x \times \ldots$ to m factors$) \times (x \times x \times \ldots$ to n factors$)$

$= x \times x \times \ldots$ to $(m+n)$ factors

$= x^{m+n}$.

235. Law II. If $m > n$, $x^m \div x^n = x^{m-n}$.

For, since $m > n$, $m - n$ is *positive.*

$\therefore$ in Law I., writing $m-n$ instead of m, we have

$$x^{m-n} \times x^n = x^{m-n+n} = x^m.$$

Dividing both sides by x^n, we have

$$x^{m-n} = x^m \div x^n.$$

236. Law III. $(x^m)^n = x^{m \times n}$.

For $(x^m)^n$ means $x^m \times x^m \times \ldots$ to n factors

$= x^{m+m+\ldots \text{ to } n \text{ terms}}$ by Law I.

$= x^{m \times n}$.

237. Law IV. If m is divisible by n, $\sqrt[n]{(x^m)} = x^{m \div n}$.

For, since m is divisible by n, $m \div n$ is an *integer.*

$\therefore$ in Law III., writing $m \div n$ instead of m, we have

$$(x^{m \div n})^n = x^{m \div n \times n} = x^m.$$

Taking the n^{th} root of both sides

$$x^{m \div n} = \sqrt[n]{(x^m)}.$$

238. The above laws may be thus expressed:

I. *Multiplying* powers is performed by *adding* indices.

II. *Dividing* powers is performed by *subtracting* indices.

III. *Power-raising* of a power is performed by *multiplying* indices.

IV. *Root-taking* of a power is performed by *dividing* indices.

Extension of Index Notation.

239. Each of the two *inverse* laws, viz., II. and IV., are proved under a condition. This condition is equivalent to the statement that the index on the right-hand side shall be *positive* (Law II.) and *integral* (Law IV.). Otherwise the right-hand side would be meaningless, for we have only defined the symbol x^m, when m is a *positive integer*.

It is convenient, however, to be able to use these formulæ for any case whatever. To do this we have only to interpret negative or fractional indices so that Laws II. and IV. shall hold universally. That is, for *all* positive integral values of m and n,

x^{m-n} shall equal $x^m \div x^n$, according to Law II.,

and $x^{\frac{m}{n}}$ shall equal $\sqrt[n]{(x^m)}$, according to Law IV.

In the formula of Law II., then

First, let $n = m$, then

x^{m-m} must equal $x^m \div x^m$,

i.e. x^0 must equal 1 ..... (1).

Secondly, let $m = 0$, then

x^{0-n} must equal $x^0 \div x^n$,

i.e. x^{-n} must equal $\frac{1}{x^n}$ ..... (2).

Thirdly, in the formula of Law IV., let m be *not* divisible by n, then

$x^{\frac{m}{n}}$ must equal $\sqrt[n]{(x^m)}$ ..... (3).

240. The interpretations (1), (2), (3) of the last article, which are of the greatest importance, thus enable us to use the four index Laws for all positive integral values of m and n. It only remains to show that, with the interpretation of negative and fractional indices given in the last article, the same laws hold for *all* values of m and n whatever. This may be shown by use of the preceding articles. Thus, e.g.

To prove $x^{\frac{a}{b}} \times x^{\frac{c}{d}} = x^{\frac{a}{b}+\frac{c}{d}}$.

$x^{\frac{a}{b}}$ means $\sqrt[b]{x^a}$; $\therefore (x^{\frac{a}{b}})^b = x^a$; $\therefore (x^{\frac{a}{b}})^{bd} = x^{ad}$ (Law III.),

$$\therefore x^{\frac{a}{b}} = \sqrt[bd]{(x^{ad})}. \qquad \text{Similarly } x^{\frac{c}{d}} = \sqrt[bd]{(x^{bc})},$$

$$\therefore x^{\frac{a}{b}} \times x^{\frac{c}{d}} = \sqrt[bd]{(x^{ad})} \times \sqrt[bd]{(x^{bc})} = \sqrt[bd]{(x^{ad} \times x^{bc})} \text{ by Art. 233, (3)}$$

$$= \sqrt[bd]{(x^{ad+bc})} \quad \text{(Law I.)}$$

$$= x^{\frac{ad+bc}{bd}} = x^{\frac{a}{b}+\frac{c}{d}}.$$

Example 1. Write down a series of powers of x beginning with x^3, in which each power is derived from the preceding by dividing by x.

$$\frac{x^3}{x} = x^2; \quad \frac{x^2}{x} = x^1; \quad \frac{x^1}{x} = 1 = x^0; \quad \frac{x^0}{x} = \frac{1}{x} = x^{-1}; \quad \frac{x^{-1}}{x} = \frac{1}{x^2} = x^{-2};$$

$$\frac{x^{-2}}{x} = \frac{1}{x^3} = x^{-3}, \text{ \&c.}$$

Thus we see that *negative* indices simply carry out the rule that to *divide* by x is equivalent to *subtracting* 1 from the *index* of x.

Example 2. Interpret $x^{\frac{1}{2}}$, $x^{\frac{2}{3}}$, $x^{\frac{3}{4}}$.

$(x^{\frac{1}{2}})^2$ must equal $x^{\frac{1}{2}\times 2}$, i.e. x^1, i.e. x.

$$\therefore x^{\frac{1}{2}} = \sqrt{x}.$$

$(x^{\frac{2}{3}})^3$ must equal $x^{\frac{2}{3}\times 3}$, i.e. x^2,

$$\therefore x^{\frac{2}{3}} = \sqrt[3]{x^2}.$$

Similarly $$x^{\frac{3}{4}} = \sqrt[4]{x^3}.$$

§ 2. Logarithmic Notation.

241. The equations

$$x^m = P \text{ and } x = \sqrt[m]{P}$$

are two different ways of stating the same relation between x, m, and P. In the first, P stands by itself; in the second, x stands by itself. There is a third way of expressing this same relation, *in which m stands by itself.* Thus

If $x^m = P$, m is called the logarithm of P to the base x; and we write $m = \log_x P$ (the base being written below).

Def. A **logarithm** of a given number to any base is the **index** of that power of the base, which is equal to the given number.

It should be remembered, then, that a *logarithm* is simply an *index* made to stand by itself. Hence the *laws of logarithms* are simply the *laws of indices* expressed in different notation.

The equations

$$x^m = P \text{ and } m = \log_x P$$

have the same meaning: and *either one of them may be substituted for the other.*

The student should examine the truth of the statement

$$x^{\log_x P} = P.$$

Example 1. What is the logarithm of 8 to the base 2?

This means to what power must 2 be raised to give 8. Now $2^3 = 8$; $\therefore$ 3 is the logarithm of 8 to the base 2; or $3 = \log_2 8$.

Example 2. Write down the values of $\log_3 81$, $\log_{10} 10000$, $\log_4 64$.

Since	$3^4 = 81$,	$\therefore 4 = \log_3 81$.
Since	$10^4 = 10000$,	$\therefore 4 = \log_{10} 10000$.
Since	$4^3 = 64$,	$\therefore 3 = \log_4 64$.

242. In using the relation $x^m = P$ or $m = \log_x P$, we shall assume for the present that x and P (i.e. the *base* and *power*) are both *positive*, and that x is not equal to 1.

243. *To show that there is only* one *positive or negative* value of the logarithm of a* positive *power to a* positive *base (not equal to* 1).

If possible let m and n be two different logarithms of P to the base x: so that

$$P = x^m = x^n.$$

Then, dividing by x^n, $\quad x^{m-n} = 1.$

Now m and n being positive or negative, $m - n$ must be positive, negative, or zero.

But, if x is not equal to 1, no positive or negative power of x can be equal to 1. For

If $x > 1$, positive powers of x are > 1; negative powers are < 1.

If $x < 1$, positive powers of x are < 1; negative powers are > 1.

$\therefore$ $m - n$ must be zero. $\quad \therefore$ $m = n$.

i.e. the two logarithms supposed to be different are not different.

Cor. In the same manner it may be shown that, if $x > 1$, the logarithms of numbers to the base x *increase as the numbers increase:* but, if $x < 1$, the logarithms of numbers to the base x *decrease as the numbers increase.*

244. In order to prove any formula involving logarithms, the student has simply to translate from the logarithmic language with which he is unfamiliar into the index language with which he is more familiar.

Thus he has only to remember that the two equations

$$x^m = P \text{ and } m = \log_x P$$

are precisely equivalent statements in the two different languages of indices and of logarithms.

* Whether there are logarithms which are neither positive nor negative is not here discussed.

245. *To prove* $\log_x 1 = 0$ *and* $\log_x x = 1$.

(1) Let $\log_x 1 = a$; *this means* $x^a = 1$.

But $x^0 = 1$, $\therefore\ a = 0$. Q.E.D.

(2) Let $\log_x x = b$; *this means* $x^b = x$.

But $x^1 = x$, $\therefore\ b = 1$. Q.E.D.

Example. Show that $\log_x \frac{1}{a} = -\log_x a$.

Let $\log_x a = m$; *this means* $x^m = a$.

But $\because a = x^m$, $\therefore \frac{1}{a} = \frac{1}{x^m} = x^{-m}$.

$\therefore \log_x \frac{1}{a} = -m = -\log_x a$. Q. E. D.

The Logarithmic Laws involving the same *base.*

246. Law I. $\log_x (m \times n) = \log_x m + \log_x n$.

Law II. $\log_x (m \div n) = \log_x m - \log_x n$.

Law III. $\log_x (m^n) = (\log_x m) \times n$.

Law IV. $\log_x (\sqrt[n]{m}) = (\log_x m) \div n$.

These laws are respectively equivalent to those of Art. 238, but expressed in the language of logarithms.

247. *To prove that*

$\log_x (m \times n) = \log_x m + \log_x n$. Law I.

$\log_x (m \div n) = \log_x m - \log_x n$. Law II.

Let $\log_x m = a$; *this means* $x^a = m$.

Let $\log_x n = b$; *this means* $x^b = n$.

Now $m \times n$ (i.e. $x^a \times x^b$) $= x^{a+b}$ by Index Law I.

And $m \div n$ (i.e. $x^a \div x^b$) $= x^{a-b}$ by Index Law II.

Expressing these equations in logarithmic language

$\log_x (m \times n) = a + b$, i.e. $\log_x m + \log_x n$.

$\log_x (m \div n) = a - b$, i.e. $\log_x m - \log_x n$.

248. *To prove that*

$$\log_x (m^n) = (\log_x m) \times n. \quad \text{Law III.}$$
$$\log_x (\sqrt[n]{m}) = (\log_x m) \div n. \quad \text{Law IV.}$$

Let $\log_x m = a$; *this means* $x^a = m$.

Now m^n, i.e. $(x^a)^n$, $= x^{a \times n}$ by Index Law III.

And $\sqrt[n]{m}$, i.e. $\sqrt[n]{(x^a)}$, $= x^{a \div n}$ by Index Law IV.

Expressing these equations in Logarithmic language,

$$\log_x (m^n) = a \times n, \text{ i.e. } (\log_x m) \times n.$$
$$\log_x (\sqrt[n]{m}) = a \div n, \text{ i.e. } (\log_x m) \div n.$$

249. The four laws just proved show that in each case the operation to be performed upon the *indices* or *logarithms* is arithmetically *simpler* than that to be performed on the *powers*.

Thus for *multiplication* of powers we substitute *addition* of logarithms;

for *division* of powers, *subtraction* of logarithms;

for *power-raising* of powers, *multiplication* of logarithms;

for *root-taking* of powers, *division* of logarithms.

250. In this simplification of arithmetical operations consists the value of Logarithms. Thus tables are published giving the logarithms of numbers to the base 10.

Suppose we had to find correctly to four decimal places the product $3{\cdot}4764 \times 7{\cdot}6819$.

From the tables we should find that

$$\log_{10} 3{\cdot}4764 = {\cdot}5411297 \text{ and } \log_{10} 7{\cdot}6819 = {\cdot}8854686.$$

Now (by Law I.)

$$\log_{10} (3{\cdot}4764 \times 7{\cdot}6819) = \log_{10} 3{\cdot}4764 + \log_{10} 7{\cdot}6819 = 1{\cdot}4265983$$

by simple *addition* of the logarithms.

Again referring to the tables, we should find that $1{\cdot}4265983$ is the logarithm of $26{\cdot}7053$.

Hence $26{\cdot}7053$ is the required product.

Logarithms to different bases.

251. *To show that* $\log_b a \times \log_a b = 1$.

Let $\log_b a = x$; *this means* $b^x = a$.

Let $\log_a b = y$; *this means* $a^y = b$.

From these two equations eliminate b, by raising the latter to the x^{th} power. Thus

$$a^{xy} = b^x = a^1, \qquad \therefore\ xy = 1,$$

$$\text{i.e. } \log_b a \times \log_a b = 1.$$

252. *To show that* $\log_a b \times \log_b c = \log_a c$.

Let $\log_a b = x$; *this means* $a^x = b$.

Let $\log_b c = y$; *this means* $b^y = c$.

From these two equations eliminate b, by raising the former to the y^{th} power. Thus

$$a^{xy} = b^y = c, \qquad \therefore\ xy = \log_a c,$$

$$\text{i.e. } \log_a b \times \log_b c = \log_a c.$$

253. *To transform from one to another system of logarithms.*

By the last article, $\log_b c = \dfrac{\log_a c}{\log_a b}$.

Hence, if we have given a system of logarithms to the base a, to find the logarithm of any number c to a new base b, we must divide the given logarithm of c in that system by the logarithm of the new base in that system.

Thus the logarithm of *any* number in the given system has to be divided by the *same* quantity $\log_a b$, in order to find its logarithm to base b.

The constant multiplier $\dfrac{1}{\log_a b}$ is called the **Modulus** of transformation from base a to base b.

§ 3. Decimal Notation and Decimal Logarithms.

254. In the decimal notation we express any number by means of the ten digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 which, *when standing alone*, stand for zero, one, two, three, &c., respectively. The 'decimal point' is placed after a digit, when it is necessary to indicate that it has its *natural* value. When digits are placed together, their value depends on their relative position. The 'units' place' being immediately before the 'decimal point', each removal of a digit to the *left raises* its value ten-fold; and hence each removal to the *right lowers* its value ten-fold. The number of removes from the units' place indicates the power of ten by which the digit has been multiplied. Digits to the *left* of the units' place correspond to the *positive*, those to the right correspond to the *negative* powers of ten. The units' place—not the decimal point—thus takes the central position.

Thus 98765·432 means

$$9.10^4 + 8.10^3 + 7.10^2 + 6.10^1 + 5.10^0 + 4.10^{-1} + 3.10^{-2} + 2.10^{-3}.$$

255. In denoting any number in this way, a cypher, which has no figures on one side of it except cyphers, is called *Insignificant*. The other figures are called *Significant*.

256. Since our system of notation has *ten* for its base, it is extremely convenient to use *ten* for the base of our logarithms. Thus, writing the minus sign *over* the negative logarithms,

$$10^0 = 1, \quad \therefore \log_{10} 1 = 0,$$

$10^1 = 10, \quad \therefore \log_{10} 10 = 1,$	$10^{-1} = 0{\cdot}1, \quad \therefore \log_{10} 0{\cdot}1 = \overline{1},$
$10^2 = 100, \quad \therefore \log_{10} 100 = 2,$	$10^{-2} = 0{\cdot}01, \quad \therefore \log_{10} 0{\cdot}01 = \overline{2},$
$10^3 = 1000, \quad \therefore \log_{10} 1000 = 3,$	$10^{-3} = 0{\cdot}001, \quad \therefore \log_{10} 0{\cdot}001 = \overline{3},$

and so on.

Any integral power of 10 is expressed by a unit and a number of zeros. Its logarithm is equal to the *number of removes of the unit from the units' place.* [Or if, as above, we fill in the units' place with a zero, *the logarithm of any integral power of ten is equal to the number of zeros by which that power is expressed.*] The logarithm is *positive or negative*, according as the removal of the unit is *backwards or forwards.*

257. The logarithm of any number, which is not an integral power of 10, will be *fractional;* and, if expressed in decimal notation, will contain an integral and a decimal part.

The method of calculating the value of logarithms will be explained in a later chapter.

The logarithms to the base 10 of all rational numbers, except those expressed by unity and zeros, are *irrational.*

Thus the logarithm of 3 to the base 10 is ·4771213 correctly to 7 places of decimals.

This means that $10^{\cdot 4771213} = 3$ approximately,

i.e. that $10^{4771213} = 3^{10000000}$ approximately.

The student is not recommended to test the truth of this statement by multiplying 3 by itself 10000000 times.

Negative Logarithms.

258. Since $\log_{10} 1 = 0$, the logarithms of all numbers *less* than 1 will be *less* than 0, i.e. *negative.* See Art. 243, Cor.

A negative logarithm, containing an integral and decimal part, is conveniently expressed in a form in which the *integral* part alone is *negative.*

Thus

$$-(3\cdot12564) = -3 - \cdot12564 = -4 + 1 - \cdot12564 = -4 + \cdot87436.$$

This is written $\bar{4}\cdot87436$: the minus sign being written *over* the integer in order to indicate that the integer alone is negative.

259. DEF. 1. The decimal part, *expressed positively,* of a logarithm is called its **Mantissa.**

DEF. 2. The integral part, *found after expressing the mantissa positively,* of a logarithm is called its **Characteristic.**

Thus the logarithm of ·00074879 is $-3\cdot12564$. But the mantissa of this logarithm is *not* $-\cdot12564$, nor is its characteristic -3. We first express it in the form $\bar{4}\cdot87436$ in which the mantissa is positive. Thus the required mantissa is ·87436 and the characteristic is -4.

260. The special convenience of using 10 as the base of Logarithms will be shown when we have proved two rules relating respectively to the *Characteristic* and *Mantissa.*

The Rule of the Characteristic.

261. The characteristic of the logarithm to the base 10 of any number expressed in decimal notation, may be seen by inspection from the *position of the first significant figure* with relation to the units' place in the given number. Thus:

The characteristic is equal to the number of removes of the first significant figure from the units' place:—being positive or negative according as the removal is backwards or forwards.

For, let s be the first significant figure: and n the number of its removes from the units' place.

Then the position of s gives it the value $s.10^n$.

Hence the given number is $> 10^n$ and $< 10^{n+1}$.

$\therefore$ its logarithm to base 10 $> n$ and $< n+1$.

For, by Art. 243, Cor., since $10 > 1$, the logarithms of numbers to the base 10 increase as the numbers increase.

But the mantissa is always *a positive quantity* < 1.

Hence the characteristic of the logarithm is n.

Examples. The characteristic of the logarithm of 32572 is $+4$ (the first significant figure 3 being *four* places to the *left* of the units' place).

For $32572 > 10000$ and < 100000; i.e. $> 10^4$ but $< 10^5$.

The characteristic of the logarithm of ·00032572 is -4 (the figure 3 being *four* places to the *right* of the units' place).

For $\cdot00032572 > \cdot0001$ and $< \cdot001$; i.e. $> 10^{-4}$ but $< 10^{-3}$.

The characteristic of the logarithm of 325·72 is $+2$ (the figure 3 being *two* places to the *left* of the units' place).

For $325\cdot72 > 100$ but < 1000, i.e. $> 10^2$ but $< 10^3$.

The Rule of the Mantissa.

262. *The mantissæ of the logarithms of two numbers are the same, if the numbers, expressed in decimal notation, have the same series of significant figures.*

For such numbers differ only in the position of the decimal point. The larger is, therefore, obtained from the smaller by multiplying it by some *positive integral power of* 10.

Let m be the larger, and n the smaller of the two numbers.

Then $m = 10^r \times n$, where r is some integer.

$\therefore \log_{10} m = \log_{10} 10^r + \log_{10} n$ (Law I.) $= r + \log_{10} n$.

Now, since the mantissæ of all logarithms are written positive, the addition of the integer r will only affect the characteristic.

Hence

the mantissa of $\log_{10} m$ = *the mantissa of* $\log_{10} n$.

Examples. The logarithm of 1075·06 is 3·0314327 (the characteristic being seen by the rule of the characteristic).

Now $107506 = 1075{\cdot}06 \times 100$,
$\therefore \log 107506 = \log 1075{\cdot}06 + 2 = 5{\cdot}0314327$;

and $1{\cdot}07506 = 1075{\cdot}06 \div 1000$,
$\therefore \log 1{\cdot}07506 = \log 1075{\cdot}06 - 3 = {\cdot}0314327$;

and ${\cdot}00107506 = 1075{\cdot}06 \div 1000000$,
$\therefore \log {\cdot}00107506 = \log 1075{\cdot}06 - 6 = \bar{3}{\cdot}0314327$.

The student should observe that each of the above characteristics follows the characteristic rule.

263. The two rules above proved make it unnecessary to publish in the tables either

(1) The *characteristic* of the logarithm of any given number, for *this* can be seen by inspection of the given number: or

(2) The *position of the decimal point* in the given number, for *this* does not affect the mantissa of the logarithm which is alone tabulated.

Hence what we find in the table is e.g.

No = 59543. Log = 7748307.

This means that

$\log 5{\cdot}9543 = {\cdot}7748307$; $\log 595{\cdot}43 = 2{\cdot}7748307$;
$\log {\cdot}0059543 = \bar{3}{\cdot}7748307$, &c. &c.

Hence a *single* reference in the table gives us the logarithms of an *indefinite number* of numbers.

§ 4. The Use of Mathematical Tables.

264. *To explain the mode of operating upon numbers whose integral part is* negative *and decimal part* positive.

An example in each of the operations of addition, subtraction, multiplication, and division will suffice.

To	$\bar{7}\cdot 432196$
Add	$\bar{5}\cdot 943205$
From	$\bar{\bar{11}}\cdot 375401$
Take	$\bar{5}\cdot 943205$
	$\bar{7}\cdot 432196 \times 5$
	$\bar{33}\cdot 160980 \div 5$
	$\bar{7}\cdot 432196$

Addition. Add the decimal parts in the ordinary way. When we reach the integral parts which are negative, we have

$$+1-7-5=-11.$$

Subtraction. Subtract the decimal parts in the ordinary way. When we reach the integral parts which are negative, we have

$$-11-(-5)-1=-7.$$

Multiplication. Multiply the decimal part in the ordinary way. When we reach the integral part which is negative, we have

$$(-7\times 5)+2=-35+2=-33.$$

Division. Find the multiple of the divisor next *higher* (instead of next *lower*) than the integral part of the dividend. Thus

$$-33\div 5=(-35+2)\div 5=-7+(2\div 5).$$

Then proceed as in ordinary division.

265. *To explain the method, when the logarithms of certain numbers are* given, *of finding the logarithms of* other *numbers* connected with those given.

[When the base is not indicated, it is understood here that the base is 10.]

The logarithms of different numbers to the same base are connected by the laws given in Art. 246.

It is always necessary to resolve a given number into its factors in order to find what numbers are logarithmically connected with the given number.

Thus:

$6 = 3 \times 2$; $150 = 3 \times 2 \times 5 \times 5 = 3 \,.\, 2 \,.\, 5^2$; $1260 = 2^2 \,.\, 3^2 \,.\, 5 \,.\, 7$.

The student should at once accustom himself to resolve any number in this way into its prime factors.

Suppose then that we have given $\log 2 = \cdot 3010300$ and $\log 3 = \cdot 4771213$, we can then find the logarithm of any number of the form $2^m \,.\, 3^n$.

For example:—

$$\begin{aligned}
\text{Log } 6 &= \log(3 \times 2) = \log 3 + \log 2 \text{ (by Law I.)} \\
&= \cdot 3010300 + \cdot 4771213 = \cdot 7781513, \\
\log 108 &= \log(27 \times 4) = \log(3^3 \times 2^2) = \log 3^3 + \log 2^2 \\
&= 3 \log 3 + 2 \log 2 \text{ (by Law III.)} \\
&= 1 \cdot 4313639 + \cdot 6020600 = 2 \cdot 0334239, \\
\log \tfrac{1}{2} &= \log 2^{-1} = -\log 2 \text{ (by Law III.)} \\
&= -\cdot 3010300 = -1 + \cdot 6989700 = \bar{1} \cdot 6989700, \\
\log \frac{9}{32} &= \log \frac{3^2}{2^5} = \log 3^2 - \log 2^5 \text{ (by Law II.)} \\
&= 2 \log 3 - 5 \log 2 \text{ (by Law III.)} \\
&= \cdot 9542426 - 1 \cdot 5051500 = \bar{1} \cdot 4490926, \\
\log (\sqrt[3]{6} \times \sqrt[5]{12}) &= \log \sqrt[3]{6} + \log \sqrt[5]{12} = \tfrac{1}{3} \log 6 + \tfrac{1}{5} \log 12 \text{ (by Law IV.)} \\
&= \tfrac{1}{3}(\log 3 + \log 2) + \tfrac{1}{5}(\log 3 + 2 \log 2) \\
&= \cdot 1590404 + \cdot 1003433 + \cdot 0954242 + \cdot 1204120 \\
&= \cdot 4752199.
\end{aligned}$$

266. The factors of 10—the base itself of our logarithms—are 2 and 5.

Thus, when either $\log 2$ or $\log 5$ is given, we may find the other: for

$$\log 5 = \log(10 \div 2) = \log 10 - \log 2 = 1 - \log 2.$$
$$\log 2 = \log(10 \div 5) = \log 10 - \log 5 = 1 - \log 5.$$

It is important to remember this.

Example. Given $\log 2 = \cdot 3010300$, find $\log 50$.

$$\begin{aligned}
\log 50 &= \log 5 + \log 10 = \log 10 - \log 2 + \log 10 \\
&= 2 \log 10 - \cdot 3010300 = 1 \cdot 6989700.
\end{aligned}$$

The Principles of Approximate Calculations.

267. When a number is given to (say) 7 decimal places, it is generally understood that the value given is *nearer* to the true value than any other number containing the *same* number of decimal places.

Thus, when we say $\log 3 = \cdot 4771213$, what is meant is that $\log 3$ is nearer to $\cdot 4771213$ than to $\cdot 4771212$ or $\cdot 4771214$; in other words it lies *between* $\cdot 47712125$ and $\cdot 47712135$.

Hence, the error involved in taking a number calculated to 7 decimal places is less than $\cdot 00000005$, i.e. less than $\frac{1}{2}$ of 10^{-7}.

When we calculate a number by *adding* or *subtracting* such approximate numbers together, we *add* together the *possible* errors.

Hence the *actual* error in the resulting sum may be greater than in any of the numbers added or subtracted.

For example $\log 3$ is really a very little greater than $\cdot 47712125$; hence (on examining a table of logarithms) we shall find that $\log 9$, i.e. $2 \log 3$, is (not $\cdot 9542426$ but) $\cdot 9542425$; that $\log 81$, i.e. $4 \log 3$, is $1 \cdot 9084850$; that $\log 729$, i.e. $6 \log 3$, is $2 \cdot 8627275$; and so on. Thus the error involved in calculating $\log 729$ from $\log 3$ is increased 6 fold and is about 3×10^{-7} instead of $\frac{1}{2}$ of 10^{-7}. (See Art. 34.)

268. It is sometimes convenient to drop the last figure in an approximately calculated number.

In such a case, the principle explained in the last article shows that the last figure *retained* should be increased by unity, *if the figure dropped is greater than* 5, or *if the figure dropped is equal to* 5 *and the true value exceeds the given value.*

Thus, dropping the last figure of $4 \cdot 321976$, we have $4 \cdot 32198$; for this is clearly *nearer* to the true value than $4 \cdot 32197$. Again, dropping the last figure of $4 \cdot 321975$, we should write $4 \cdot 32198$ or $4 \cdot 32197$ according as the approximation $4 \cdot 321975$ is *less* or *greater* than the true value.

269. The student is advised to make use of some book of Mathematical Tables: such as Chambers's.

The following lists of corresponding numbers will be found:

(1) *The mantissæ of logarithms, calculated to 7 places of decimals,* which correspond to the 100,000 *numbers between* 0 *and* 100,000, *successively increasing by unity.*

(2) *The trigonometrical ratios, calculated to 7 places of decimals,* which correspond to the 5400 *angles between* 0° *and* 90°, *successively increasing by* 1 *minute.*

(3) *The logarithms* (*increased by* 10), *calculated to 7 places of decimals,* which correspond *to the ratios of these angles.*

270. If it is required to find a number corresponding to any number given in the tables, mere reference to these tables is sufficient. But if it is required to find a number corresponding to some number *intermediate between two consecutive numbers in the table,* we must 'interpolate' according to the following rule; the foundation and limits of which will be explained in a later chapter.

Rule of Proportional Differences.

271. The small differences between the values of one varying quantity are approximately proportional to the small differences between the *corresponding* values of another quantity which varies with the first.

Thus, if L and N are the values corresponding to *successive* values l and n given in the table, and if we have to find the value M corresponding to m, which lies *between* l and n, we have the equation

$$\frac{M-L}{N-L} = \frac{m-l}{n-l}.$$

Thus the difference $M-L$, which has to be added to the given value L to find M,

$$= \frac{m-l}{n-l} \times (N-L).$$

Example 1. From the tables we find that

$$\log 35745 = 4{\cdot}5532153 \text{ and } \log 35746 = 4{\cdot}5532275.$$

The difference of these logarithms is ·0000122.

To find, then, log 35745·73, we have by above rule

$$\frac{\log 35745{\cdot}73 - \log 35745}{\log 35746 - \log 35745} = \frac{35745{\cdot}73 - 35745}{35746 - 35745},$$

$$\therefore \log 35745{\cdot}73 - \log 35745 = \frac{73}{100} \text{ of } {\cdot}0000122 = {\cdot}0000089,$$

$$\therefore \log 35745{\cdot}73 = \log 35745 + {\cdot}0000089 = 4{\cdot}5532242.$$

Example 2. From the same data to find conversely the number whose logarithm is 4·5532242, i.e. the value of $10^{4{\cdot}5532242}$.

We have

$$\frac{10^{4{\cdot}5532242} - 10^{4{\cdot}5532153}}{10^{4{\cdot}5532275} - 10^{4{\cdot}5532153}} = \frac{4{\cdot}5532242 - 4{\cdot}5532153}{4{\cdot}5532275 - 4{\cdot}5532153},$$

$$\therefore 10^{4{\cdot}5532242} - 35745 = \frac{89}{122} = {\cdot}73,$$

$$\therefore 10^{4{\cdot}5532242} = 35745{\cdot}73.$$

This process of interpolation is aided by the introduction of

Subsidiary Tables.

Thus, at the side of the main table, a column headed **Diff.** will be found.

The difference of the logarithms, corresponding to a difference of 1 in the numbers, *at any particular part of the table*, is here tabulated.

And below this difference is tabulated the differences corresponding to 1, 2, 3, 4, 5, 6, 7, 8, 9 *tenths* respectively.

Dividing these latter differences by 10, we have those corresponding to 1, 2, 3, 4, 5, 6, 7, 8, 9 *hundredths* respectively.

Thus for the above example, we have tabulated

Diff. for 1 = 122; for 7 tenths = 85, for 3 hundredths = 3·7

$$\therefore \text{Diff. for } {\cdot}73 = 89 \text{ nearly.}$$

272. Examples in interpolation may be written in the following form :—

Given log 35745 and log 35746 find log 35745·73.

From log 35746 = 4·5532275
Take log 35745 = 4·5532153

∴ when diff. in nos. = 1, then 122 = diff. in logs,
∴ when diff. in nos. = ·73, then ·73 of 122 = diff. in logs.

[Now ·73 of 122 = ·7 of 122 + ·03 of 122 = 85 + 4 = 89,]

∴ when diff. in nos. = ·73, then 89 = diff. in logs.
Add log 35745 = 4·5532153

∴ log 35745·73 = 4·5532242

Conversely, given log 35745 and log 35746, find the number whose logarithm is 4·5532242.

From 4·5532275 = log 35746
Take 4·5532153 = log 35745

∴ when diff. in logs = 122, then 1 = diff. in nos.
From 4·5532242 = log (required number)
Take 4·5532153 = log 35745

∴ when diff. in logs = 89, then $\frac{89}{122}$ = diff. in nos.

$$\left[\text{Now } \frac{89}{122} = \frac{85+4}{122} = \cdot 73\right]$$

∴ when diff. in logs = 89, then ·73 = diff. in nos.
∴ 4·5532242 = log 35745·73.

273. In the same way we may interpolate in the tables of the *ratios* of angles, and of the *logarithms* of the ratios.

274. It is important to observe that, as the *secondary* ratios *co*sine, *co*tangent, *co*secant vary *incongruently* with the angles, their differences and the differences of their logarithms will have the *opposite* sign to the corresponding differences of the angles.

Example. To find cos 30° 13′ 2″,

$$\frac{\cos 30° 13' 2'' - \cos 30° 13'}{\cos 30° 14' - \cos 30° 13'} = \frac{\cos 30° 13' - \cos 30° 13' 2''}{\cos 30° 13' - \cos 30° 14'}.$$

By the principle of Proportional Differences, this fraction

$$= \frac{30° 13' 2'' - 30° 13'}{30° 14' - 30° 13'} = \frac{2''}{60''} = \frac{1}{30}.$$

Now $\cos 30° 13' = \cdot 8641284$

$\cos 30° 14' = \cdot 8639820$

$\therefore$ diff. in cos for $1' = \cdot 0001464$

$\therefore$ diff. in cos for $2'' = \cdot 0000049$

$\therefore \cos 30° 13' 2'' = \cdot 8641235.$

This result is obtained by *subtracting* the difference, corresponding to 2″, from cos 30° 13′.

Or we might *add* the difference, corresponding to 58″, to cos 30° 14′.

275. One half of the ratios of angles between 0 and 90°, viz. all the sines and cosines and all the tangents up to 45°, are *less than* 1. The logarithms of these ratios are, therefore, *negative*.

In order to avoid tabulating so many negative quantities, 10 *is added to the logarithms* of *all* ratios in the tables. In this way all the tabulated logarithms are reduced to *positive* numbers. [The number 4, instead of 10, would answer the same purpose: for the smallest tabulated logarithm, viz. log sin 1′, is greater than -4.]

The symbol L is used to denote a logarithm *increased by* 10.

Thus $L \sin 28° 37'$ means $10 + \log \sin 28° 37'$

$$= 10 + \bar{1}\cdot 6802877 = 9\cdot 6802877.$$

§ 5. Adaptation of formulæ to logarithmic calculation.

276. Numbers which require for notation several significant figures may be called *long numbers*.

The practical use of logarithms is to enable us to avoid having to perform any operations, involving long numbers, except *addition* and *subtraction*.

277. For logarithmic purposes an expression requires to be put into *factor-form*.

We here use the word factors to denote the parts of an expression which, being either multiplied or *divided*, make up the expression. Thus:

If $P = A \times B$; $\log P = \log A + \log B$.

If $Q = A \div B$; $\log Q = \log A - \log B$.

Strictly, of course, $\frac{1}{B}$, not B, is the *factor* in the second case.

278. A formula is best adapted to logarithmic calculation, when the factors of the required value involve only the *addition* and *subtraction* of long numbers.

The following articles will illustrate this statement.

279. To solve a right-angled triangle, whose sides a and b (containing the right-angle) are given.

I. To find c, we have $c^2 = a^2 + b^2$.

We might find a^2 and b^2 by multiplication, which would be a troublesome process if a and b were long numbers. We may use the logarithmic table to avoid this process; but not conveniently. Thus:

Suppose $a = 3456{\cdot}4$ ft. and $b = 4543{\cdot}5$ ft.

then $\log a = 3{\cdot}5386240$ and $\log b = 3{\cdot}6573905$,

$\therefore$ $2 \log a = 7{\cdot}07725$ and $2 \log b = 7{\cdot}31478$.

Now $\log 1{\cdot}1947 = {\cdot}07725$ and $\log 2{\cdot}0643 = {\cdot}31478$,

$\therefore$ $a^2 = 11947000$ and $b^2 = 20643000$,

$\therefore$ $c^2 = 32590000$ and $\log c^2 = 7{\cdot}5130844$,

$\therefore$ $\log c = 3{\cdot}7565423$ and $c = 5708{\cdot}8$.

This work requires 6 references to the table; showing that the formula $c^2 = a^2 + b^2$ is *not* well adapted for logarithmic purposes.

II. To find A, we have $\tan A = \frac{a}{b}$.

$\therefore$ $\log \tan A = \log a - \log b = \bar{1}{\cdot}8812335$,

$\therefore$ $L \tan A = 10 + \log \tan A = 9{\cdot}8812335$.

Now $9{\cdot}8813144 = L \tan 37^\circ\ 16'$

$9{\cdot}8810522 = L \tan 37^\circ\ 15'$

$\therefore$ ${\cdot}0002622 =$ diff. for $1'$

$\therefore$ ${\cdot}0001813 =$ diff. for $\frac{9}{13}$ of $60''$ i.e. $42''$

$\therefore$ $A = 37^\circ\ 15'\ 42''$ and $B = 90^\circ - A = 52^\circ\ 44'\ 18''$.

III. Having found A, we may find c as follows:

We have $\frac{c}{a} = \text{cosec } A = \text{cosec } 37° 15' 42''$,

$\therefore \log c - \log a = \log \text{cosec } 37° 15' 42''$.

Now $L \text{ cosec } 37° 15' = 10{\cdot}2180336$

$L \text{ cosec } 37° 16' = 10{\cdot}2178676$

$\therefore$ diff. for $1' = \quad {\cdot}0001660$

$\therefore$ diff. for $42'' = \frac{7}{10}$ of $\frac{166}{100000} = {\cdot}0001162$,

$\therefore L \text{ cosec } 37° 15' 42'' = 10{\cdot}2179174$,

$\therefore \log c = 3{\cdot}5386240 + {\cdot}2179174 = 3{\cdot}7565414$,

$\therefore c = 5708{\cdot}8$ as before.

280. To find the side a of a right-angled triangle, when the hypothenuse c and third side b are given.

Here $a^2 = c^2 - b^2 = (c+b)(c-b)$.

In this case the required value is expressed by factors *which involve addition and subtraction only.* Hence it is better adapted for logarithmic calculation.

Thus $2 \log a = \log (c+b) + \log (c-b)$.

Suppose $c = 5708{\cdot}8$ and $b = 4543{\cdot}5$,

$\therefore c+b = 10252{\cdot}3$ and $c-b = 1165{\cdot}3$,

$\therefore \log (c+b) = 4{\cdot}0108213$ and $\log (c-b) = 3{\cdot}0664377$,

$\therefore 2 \log a = 7{\cdot}0772590$, $\therefore \log a = 3{\cdot}5386295$,

$\therefore a = 3456{\cdot}4$.

281. To solve a triangle when two angles and a side are given. [See Arts. 163, 164.]

Here $A = 180° - B - C$; which gives A by mere *subtraction.*

Next $b = \frac{a \sin B}{\sin A}$, $\therefore \log b = \log a + \log \sin B - \log \sin A$.

This is well adapted to logarithmic calculation. Similarly c may be found.

282. To solve a triangle when the three sides are given. [See Arts. 165, 166.]

We have $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

The factor $b^2 + c^2 - a^2$ has here to be calculated by multiplications: hence this formula is *not* well adapted for logarithmic calculations. On the other hand, the formula

$$\tan \frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}} \text{ gives}$$

$$\log \tan \tfrac{1}{2}A = \tfrac{1}{2}\log(s-b) + \tfrac{1}{2}\log(s-c) - \tfrac{1}{2}\log(s-a) - \tfrac{1}{2}\log s.$$

This is well adapted.

Or we may use

$$\sin \frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}} \text{ or } \cos \frac{A}{2} = \sqrt{\left\{\frac{s(s-a)}{bc}\right\}}.$$

If *all* the angles are to be found the *tan* formula is the most convenient, because only the four logarithms of s, $s-a$, $s-b$, $s-c$ will be required for the three angles.

Example. Given $a = 35742$ yards, $b = 29813$ yards, $c = 47265$ yards, find A.

35742	56410	56410	56410
29813	35742	29813	47265
47265	20668	26597	9145
2)112820	$=s-a$	$=s-b$	$=s-c$
$s = 56410$			

$\therefore$ From the tables we find

$\log s = 4{\cdot}7513561$	$\log(s-b) = 4{\cdot}4248327$
$\log(s-a) = 4{\cdot}3152985$	$\log(s-c) = 3{\cdot}9611837$
(adding) $= 9{\cdot}0666546$	(adding) $= 8{\cdot}3860164$
$8{\cdot}3860164$	
$2)\overline{{\cdot}6806382}$	
$\overline{{\cdot}3403191} = \tfrac{1}{2}\log s + \tfrac{1}{2}\log(s-a) - \tfrac{1}{2}\log(s-b) - \tfrac{1}{2}\log(s-c).$	

$\therefore L\tan \tfrac{1}{2}A = 10 - {\cdot}3403191 = 9{\cdot}6596809.$

Now $9{\cdot}6597076 = L\tan 24^\circ\, 33'$

$9{\cdot}6593733 = L\tan 24^\circ\, 32'$

${\cdot}0003343 =$ diff. for $1'$

$\therefore$ ·0003076 = diff. for $\frac{10}{11}$ of 60″, i.e. 54″·5,

$\therefore \tfrac{1}{2}A = 24^\circ\, 32'\, 54''{\cdot}5$, $\therefore A = 49^\circ\, 5'\, 49''$.

283. To solve a triangle when two sides and the included angle are given. [See Arts. 167, 168.]

I. To find a, we have $a^2 = b^2 + c^2 - 2bc \cos A$.

This is not well adapted to logarithmic calculation. We may, however, transform it as follows:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\,(1 - 2\sin^2 \tfrac{1}{2} A), \\ &= (b-c)^2 + 4bc \sin^2 \tfrac{1}{2} A, \\ &= (b-c)^2 \left\{ 1 + \frac{4bc}{(b-c)^2} \sin^2 \frac{A}{2} \right\}. \end{aligned}$$

Now there must be some angle, θ say, whose tangent

$$= \frac{2\sqrt{(bc)}}{b-c} \sin \frac{A}{2}.$$

By the help of the tables this angle may be found. Thus since

$$\tan \theta = \frac{2\sqrt{(bc)}}{b-c} \sin \frac{A}{2}$$

$$\therefore \log \tan \theta = \log 2 + \tfrac{1}{2} \log b + \tfrac{1}{2} \log c - \log (b-c) + \log \sin \tfrac{1}{2} A.$$

Having found θ, we have

$$\begin{aligned} a^2 &= (b-c)^2 (1 + \tan^2 \theta) = (b-c)^2 \sec^2 \theta, \\ \therefore\ a &= (b-c) \sec \theta, \\ \therefore \log a &= \log (b-c) + \log \sec \theta. \end{aligned}$$

II. To find B and C, we have

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2},$$

$$\therefore \log \tan \frac{B-C}{2} = \log (b-c) - \log (b+c) + \log \cot \frac{A}{2}.$$

Having found B and C we may find a from either of the equations

$$\begin{aligned} &a = b \sin A \operatorname{cosec} B, \\ \text{or } &a = (b+c) \sin \tfrac{1}{2} A \sec \tfrac{1}{2} (B-C), \\ \text{or } &a = (b-c) \cos \tfrac{1}{2} A \operatorname{cosec} \tfrac{1}{2} (B-C). \end{aligned}$$

The two latter of these (which the student may easily prove) have the advantage over the first that they require only *two* new logarithms. Thus

$$\log a = \log (b+c) + \log \sin \tfrac{1}{2} A + \log \sec \tfrac{1}{2} (B-C).$$

Example. Given $b=723$, $c=259$, $A=35°\ 18'$, solve the triangle.

Let $\tan\theta=\dfrac{2\sqrt{(bc)}}{b-c}\sin\dfrac{A}{2}$.

Then $L\tan\theta=\log 2+\frac{1}{2}\log b+\frac{1}{2}\log c-\log(b-c)+L\sin 17°\ 39'$

$=\cdot3010300+1\cdot4295691+1\cdot2066499$

$\qquad -2\cdot6665180+9\cdot4817315$

$=9\cdot7524625,$

$\therefore\ \theta=29°\ 29'\ 23''$ by reference to the tables.

Now $a=(b-c)\sec\theta,$

$\therefore\ \log a=\log(b-c)+\log\sec\theta$

$=2\cdot6665180+\cdot0602592$

$=2\cdot7267772,$

$\therefore\ a=533\cdot0614=533\cdot06$ approximately.

Next $L\tan\frac{1}{2}(B-C)=\log(b-c)-\log(b+c)+L\cot\frac{1}{2}A$

$=2\cdot6665180-2\cdot9921115+10\cdot4973279$

$=10\cdot1717344,$

$\therefore\ \frac{1}{2}(B-C)=56°\ 2'\ 39'',$

and $\frac{1}{2}(B+C)=72°\ 21',$

$\therefore\ B=128°\ 23'\ 39''$ and $C=16°\ 18'\ 21''.$

Hence $\log a=\log(b+c)+\log\sin 17°\ 39'+\log\sec 56°\ 2'\ 39'',$

$=2\cdot9921115+\bar{1}\cdot4817315+\cdot2529351,$

$=2\cdot7267781,$

$\therefore\ a=533\cdot0625=533\cdot06$ approximately (as above).

284. To solve a triangle when two sides and a non-included angle are given. [See Arts. 169, 170.]

Here $$\sin C=\frac{c\sin B}{b},$$

$$\therefore\ L\sin C=\log c-\log b+L\sin B.$$

This formula is immediately adapted to logarithmic calculation and need not be further discussed.

EXAMPLES XI.

1. Prove from definition (the indices being positive integers):

$$\sqrt[mn]{a^{pn}} = \sqrt[m]{a^p}\,; \quad \sqrt[n]{a^{pn}} = a^p\,; \quad \sqrt[n]{(a^p)} = (\sqrt[n]{a})^p\,;$$
$$\sqrt[n]{a^p} \times \sqrt[m]{a^q} = \sqrt[mn]{a^{pm+nq}}\,; \quad \sqrt[n]{a^p} \div \sqrt[m]{a^q} = \sqrt[mn]{a^{pm-nq}}\,;$$
$$\sqrt[m]{\{(\sqrt[n]{a^p})\}^r} = \sqrt[mn]{a^{pr}}.$$

2. Interpret a^0, a^{-5}, $a^{\frac{3}{5}}$, $a^{-\frac{4}{7}}$; and write down the logarithms of each expression to the base a.

3. Find the values of 3^{-4}, $27^{\frac{4}{3}}$, $128^{-\frac{5}{7}}$, 219^0.

4. Express the identities in Ex. 1, in terms of *powers* instead of *roots* of a.

5. Express as powers of x:—

$$\sqrt[5]{x^3},\ \frac{1}{x^7},\ \frac{1}{\sqrt{x^5}},\ 1,\ \frac{x^2}{x^{\frac{1}{3}}},\ \sqrt{x^5} \times \sqrt[3]{x^4},\ \frac{\sqrt{x^7}}{\sqrt[3]{x^2}};$$

and write down the logarithms of each expression to the base x.

6. Find the values of

$\log_3 243$, $\log_7 343$, $\log_4 64$, $\log_{10} 10000$, $\log_3 81$,
$\log_{11} \sqrt[5]{11^{17}}$, $\log_3 \sqrt{27}$, $\log_2 \sqrt[5]{\frac{1}{64}}$, $\log_5 \sqrt{125}$,
$\log_{18} \frac{1}{169}$, $\log_4 2$, $\log_{27} 3$, $\log_9 \sqrt{81}$, $\log_8 \sqrt[3]{2}$,
$\log_4 \sqrt[5]{\frac{1}{2}}$, $\log_{10} \dfrac{1}{\sqrt{1000}}$, $\log_5 \dfrac{\sqrt[3]{5}}{\sqrt{125}}$,
$\log_{10} \cdot 00001$, $\log_2 \cos 60°$, $\log_2 \cos 45°$,
$\log_2 \cdot 5$, $\log_5 \cdot 04$, $\log_{32} 128$.

7. Find the characteristics of

$\log_{10} 3245$, $\log_{10} 123$, $\log_{10} 3 \cdot 45$, $\log_{10} \cdot 1234$,
$\log_{10} \cdot 001234$, $\log_{10} 57000$, $\log_3 90$, $\log_4 100$,
$\log_5 200$, $\log_6 50$, $\log_{10} (123 \times 10^9)$.

Prove the following statements: (8—23).

8. $a^{\log b} = b^{\log a}$. 9. $a^{\log bc} = a^{\log b} \cdot a^{\log c}$.

10. $a^{\log b^c} = (a^{\log b})^c$.

11. $\log_a b \times \log_b c \times \log_c a = 1$.

12. $\log_a b \times \log_c d = \log_a d \times \log_c b$.

13. $\log_a b \times \log_b c \times \log_c d \times \log_d e = \log_a e$.

14. $\log_a m^x \times \log_b n^y \times \log_c r^z = \log_b r^x \times \log_c m^y \times \log_a n^z$.

15. If $\log_a n = \log_b m$, then each $= \log_{ab} mn$.

16. If $\log_a x = \log_b y = \log_c z$, then each $= \log\ x^p y^q z^r$ to the base $a^p b^q c^r$.

17. $\log\ x$ to base $a^n = \log \sqrt[n]{x}$ to base a.

18. $2 \log \cos A = \log (1 + \sin A) + \log (1 - \sin A)$.

19. $2 \log \tan A = \log (\sec A - 1) + \log (\sec A + 1)$.

20. $\log \sin 2A = \log 2 + \log \cos A + \log \sin A$.

21. $\log \cos 2A = \log (\cos A + \sin A) + \log (\cos A - \sin A)$.

22. $\log (1 + \sin 2A) = 2 \log (\cos A + \sin A)$.

23. $\log \tan (A + 45°) = \log (1 + \tan A) - \log (1 - \tan A)$.

Logarithms to base 10.

24. Given $\log 2 = \cdot 3010300$, find the logarithms of

$$4,\ 8,\ 128,\ \sqrt[3]{2},\ 1024,\ \tfrac{1}{2},\ \frac{1}{\sqrt[5]{64}},\ \cdot 25,\ \cdot 03125,\ 200,\ \cdot 002.$$

25. Given $\log 2 = \cdot 3010300$, find the logarithms of

$$5,\ 25,\ 50,\ \cdot 2,\ \cdot 008,\ \frac{1}{\sqrt[4]{125}},\ 5000,\ \cdot 0005.$$

26. Given $\log 3 = \cdot 4771213$, find the logarithms of

$$9,\ \tfrac{1}{3},\ 30,\ 243,\ \cdot 03,\ \cdot \dot{1}.$$

27. Given $\log 2$ and $\log 3$ (as above) find the logs of

$$6,\ 18,\ 15,\ 135,\ 144,\ 750,\ \cdot \dot{2},\ \cdot 004.$$

28. Given $\log 7 = \cdot 8450980$, find the logs of

$$343,\ 343000,\ \cdot 343,\ \cdot 00343,\ 3 \cdot 43.$$

29. Given $\log 2$, $\log 3$, and $\log 7$ (as above) find the logs of

$$35,\ 210,\ 245,\ 28,\ 42,\ 63,\ 441,\ 504.$$

State in each case the possible error in the result calculated.

30. Given log 5 = ·6989700, log 11 = 1·0413927, find the logarithms of 2·2, ·36, 4·54.

31. Given log 18 = 1·2552725, log 24 = 1·3802112, find log 2 and log 3.

32. Given log 54 = 1·7323938, log 45 = 1·6532125, find log 3 and log 5.

33. Given log 48 = 1·6812412, log 75 = 1·8750613 find log 15, log 12, and log ·02.

34. Prove that

$$L \operatorname{cosec}\theta + L \sin\theta = L \cot\theta + L \tan\theta = L \cos\theta + L \sec\theta = 20.$$

35. Find the cube root of two billion, having given log 2 = ·3010300 and log 1·259921 = ·1003433.

36. Given

log π = ·4971499, log 1·3949 = ·1445431, log 4·382208 = ·6416930, find the length of the equator, the equatorial radius of the earth being about 6974500 yards.

37. Show that the characteristic of the logarithm to base 10 of a *whole* number is one less than the number of digits by which that number is denoted in the decimal system of notation.

38. Given log 2, find the number of digits in 2^{135}.

39. Find the 5th root of the product of ·00056789 and ·012345 having given that

log 56789 = 4·7542642,
log 12345 = 4·0914911,
log 931432 = 5·9691511.

On Interpolation.

40. Given log 8·4713 = ·9279501 and log 8·4714 = ·9279552; find log 8·471329, log 847·1343, and log 84713·76.

41. Given log 187·13 = 2·2721434 and log 187·14=2·2721666; find log 187136·7, log ·01871359, and log 18·71334.

42. Given ·4271776 = log 2·6741 and ·4271939 = log 2·6742; find the numbers whose logarithms are 3·4271802, 3·4271799, $\bar{1}$·4271900.

43. Given 2·5770320 = log 377·60 and ·5770435 = log 3·7761; find the numbers whose logarithms are 2·5770397, $\bar{3}$·5770409, ·5770421.

44. Given sin 38° 38′ = ·6243342 and sin 38° 39′ = ·6245614; find sin 38° 38′ 38″, and sin 38° 38′ 57″.

45. Given cot 43° 23′ = 1·0580867 and cot 43° 24′ = 1·0574704; find cot 43° 23′ 23″ and cot 43° 23′ 42″.

46. Given L tan 32° 32′ = 9·8047447 and L tan 32° 33′ = 9·8050233; find L tan 32° 32′ 32″ and L tan 32° 32′ 46″.

47. Given L cos 21° 21′ = 9·9691241 and L cos 21° 22′ = 9·9690746; find L cos 21° 21′ 21″ and L cos 21° 21′ 35″.

Solution of Triangles.

48. Given $C = 90°$, $c = 934$, $a = 257$; find b.

[log 1·191 = ·0759118, log 6·77 = ·8305887,
log 8·9794 = ·9532473, log 8·9795 = ·9532522.]

49. Given $C = 90°$, $a = 532·7$, $b = 261·9$; find A, B, c.

[log 5·327 = ·7264827, log 2·619 = ·4181355,
L tan 63° 49′ = 10·3083000, L tan 63° 50′ = 10·3086191,
L sec 63° 49′ = 10·3553204, L sec 63° 50′ = 10·3555774,
log 5·93601 = ·7734944.]

50. Given $C = 90°$, $a = 2795$, $A = 60°$; find b, c, B.

[log 2·795 = ·4463818, log 3 = ·4771213,
log 1·6136 = ·2077959, log 1·6137 = ·2078228.]

51. Given $B = 70°\ 13'$, $C = 23°\ 19'$, $a = 143$; find b.

[L sin 70° 13′ = 9·9735801, L cosec 86° 28′ = 10·0008263,
log 1·43 = ·1553360,
log 1·3481 = ·1297221, log 1·3482 = ·1297543.]

52. Given $B = 59° 16'$, $C = 45°$, $b = 2793$; find a.

[cot 59° 16′ = ·5945, log 1·5945 = ·2026245,
log 2·793 = ·4460709, log 5 = ·6989700,
log 3·1490 = ·4981727, log 3·1491 = ·4981865.]

53. Given $a = 52317$, $b = 24659$, $c = 47932$; find A.

[log 6·2454 = ·7955603, log 1·0137 = ·0059094,
log 3·7795 = ·5774343, log 1·4522 = ·1620264,
L tan 42° 57′ = 9·9688960, L tan 42° 58′ = 9·9691493.]

54. The sides of a triangle are as 9 : 11 : 13. Find the greatest angle.

[cos 80° 24′ = ·1667687, cos 80° 25′ = ·1664819.]

55. The sides of a triangle are 50, 36, 28. Find the greatest angle.

[log 19 = 1·2787536, log 29 = 1·4623980,
L tan 51° 0′ = 10·0916308, L tan 51° 1′ = 10·0918891.]

56. The sides of a triangle are as 7 : 8 : 9; find all the angles.

[L tan 24° 5′ = 9·6502809, L tan 24° 6′ = 9·6506199,
L tan 29° 12′ = 9·7473194, L tan 29° 13′ = 9·7476160,
log 2 = ·3010300.]

57. Given $b = 7235$, $c = 1592$, $A = 50°$; find B and C.

[log 5·643 = ·7515101, log 8·827 = ·9458131,
L cot 25° = 10·3313275, L tan 53° 53′ 33″ = 10·1370245.]

58. Given $b = 27$, $c = 13$, $A = 120°$; find B, C, and a.

[log 2 = ·3010300, log 3 = ·4771213,
log 7 = ·8450980, log 3·5341 = ·54828,
L tan 11° 25′ = 9·3052183, L tan 11° 26′ = 9·3058689,
L sec 11° 25′ = 10·0086793, L sec 11° 26′ = 10·0087048.]

59. Given $b = 9268$, $c = 6951$, $A = 16° 15' 38''$; find B and C.

[log 7 = ·8450980, L cot 8° 7′ 49″ = 10·8450980.]

60. Given $B = 47° 13'$, $c = 239{\cdot}7$, $b = 198{\cdot}5$; find A and C.

[log $2{\cdot}397 = {\cdot}3796680$, log $1{\cdot}985 = {\cdot}2977605$,
$L \sin 62° 24' = 9{\cdot}9475335$, $L \sin 62° 25' = 9{\cdot}9475995$,
$L \sin 47° 13' = 9{\cdot}8656531$.]

61. Given $B = 47° 13'$, $b = 239{\cdot}7$, $c = 198{\cdot}5$; find C, A, a.

[$L \sin 37° 25' = 9{\cdot}7836227$, $L \sin 37° 26' = 9{\cdot}7837878$,
$L \cos 5° 21' = 9{\cdot}9981040$, $L \cos 5° 22' = 9{\cdot}9980921$,
log $3{\cdot}2517 = {\cdot}5121105$, log $3{\cdot}2518 = {\cdot}5121238$.
See last Example.]

62. Given $B = 123° 27'$, $b = 349{\cdot}87$, $c = 37{\cdot}925$; find C, A, a.

[log $3{\cdot}4987 = {\cdot}5439067$, log $3{\cdot}7925 = {\cdot}5789256$,
$L \cos 33° 27' = 9{\cdot}9213572$, log $3{\cdot}275311 = {\cdot}5152526$,
$L \sin 5° 11' = 8{\cdot}9558940$, $L \sin 5° 12' = 8{\cdot}9572843$,
$L \sin 51° 21' = 9{\cdot}8926375$, $L \sin 51° 22' = 9{\cdot}8927385$.]

[Reference to tables will be required for Ex. 63—67.]

63. A tower 132 feet high throws a shadow 66 feet long upon the horizontal plane upon which it stands. Find the Sun's altitude.

64. The elevation of a balloon, at a height 3000 ft. from the ground, is $43° 17'$; find its distance from the spectator.

65. The angles of depression of the two banks of a river, 243 feet wide, (as observed from a point in a vertical plane perpendicular to the length of the river,) are $23° 17'$ and $18° 15'$; find the vertical height of the point of observation.

66. From the bank of a river, the top of a tower on the opposite bank is at an elevation $54°$; 35 feet further away its elevation is $49°$; find the breadth of the river (the two points of observation and the tower being in a line perpendicular to the length of the river).

67. The angle of elevation of a hill from a point due north of it is $53° 18' 27''$, and from another point due west of the

former and distant 430·306 feet from it the elevation is 49° 17′ 18″. Find the height of the hill.

68. Given two sides and an angle (b, c, A) of a triangle, show that the third side $a = (b + c) \cos \theta$, where

$$\sin \theta = \frac{2\sqrt{(bc)}}{b + c} \cos \frac{A}{2}.$$

69. Given the sides of a triangle, show that the angles may be found from the equations

$$\cos \frac{A - B}{2} = \frac{(a + b) \sin \theta}{2\sqrt{(ab)}}, \qquad \sin \frac{C}{2} = \frac{c \sin \theta}{2\sqrt{(ab)}},$$

where $a - b = c \cos \theta$.

70. Given the sides of a triangle, show that the angles may be found from the equations

$$\log \cos B = \log x - \log c, \qquad \log \cos C = \log y - \log b,$$

where x and y are found from the equations

$$x + y = a, \quad \log (x - y) = \log (c + b) + \log (c - b) - \log a.$$

71. Show, by examining in the tables the logarithms of the numbers 1, 2, 3, 4, 5, &c., and the logarithms of the sines of the angles 1′, 2′, 3′, 4′, 5′, &c., that the sines of small angles are approximately to one another as the angles.

CHAPTER XII.

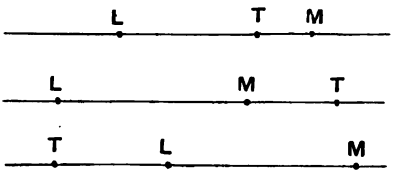
RATIOS OF ANGLES UNLIMITED IN SIGN AND MAGNITUDE.

§ 1. Directed Magnitudes.

285. Let L and M be two points in a line: and let LM stand for the *length* of the part of the line between them.

Let T be any other point in the same line.

Then we may express LM in terms of LT and TM.



Thus

(1) If T is between L and M, $LM = LT + TM$ .....(1),

(2) If T is outside LM, beyond M, $LM = LT - MT$ ...(2),

(3) If T is outside LM, beyond L, $LM = TM - TL$ ...(3).

If we do not know where T is situated with regard to L and M, we do not know which of these three formulæ to adopt. But it would be clearly advantageous to be able to use *the same equation for all three cases.* We observe, then, that

MT has to be *subtracted,* when T is on the side of M, *opposite to* L: and that

TL has to be *subtracted,* when T is on the side of L, *opposite to* M: and that

Otherwise MT and TL have to be *added.*

Directed Lengths.

286. Let us, then, use a bracketed symbol, such as (LM), to indicate—not merely the distance LM—but also the *direction* in which that distance is regarded as having been described.

Such a symbol as (LM) or (ML) may be said to represent a **directed length.** Thus

(LM) represents the line LM, regarded as drawn from L to M;

(ML) represents the line LM, regarded as drawn from M to L.

287. The use of these symbols representing directed lengths is explained by the following **rules of interpretation.**

To interpret any expression in which directed lengths in the same line *are connected by the signs* + *or* −.

Rule I. When the lengths are all directed *the same way*, the signs must be understood in their Arithmetic sense.

Rule II. Any directed length may be *reversed*, if the sign preceding it is *changed.*

To obtain the final interpretation, therefore, we must reduce all the directed lengths to lengths directed the same way, by reversing and changing the preceding signs where necessary.

288. From the rules of the preceding article, it will be seen that we may write always in Art. 285

$$(LM) = (LT) + (TM) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

For, in fig. 1, (LT), (TM), and (LM) are *directed the same way*, and hence + has to be interpreted as arithmetic addition.

In fig. 2, (TM) is directed in the opposite way to (LT) and (LM), and $\therefore + (TM)$ becomes $- (MT)$.

In fig. 3, (LT) is directed in the opposite way to (TM) and (LM), and $\therefore + (LT)$ becomes $- (TL)$.

289. The formula $(LM) = (LT) + (TM)$ is of the highest importance. To give it a familiar aspect, we may regard (LM) as a *step* from L to M. Thus, if we have to go from L to M, we may always go from L to T and then from T to M.

This will be true, even if T is *not in the line* LM. But in this case LTM would form a (rectilineal or curvilinear) triangle, and we should not (in general) be able to give any arithmetic meaning to the sign +, for neither the addition nor the subtraction of the lengths of two sides of a triangle would give us (in general) the length of the third side.

If L, T, M are in the same line (straight or curved) we may always give an *Arithmetic Interpretation* to the equation $(LM) = (LT) + (TM)$, by reversing and changing the sign of either (LT) or (TM), if it is in the direction opposite to (LM).

290. The student should observe that with the same rules of interpretation, the equations

$$(LM) = (LT) - (MT) \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

and

$$(LM) = (TM) - (TL) \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

are always true, if L, T, M are in a line.

In fact, by Rule 2,

$$+(TM) = -(MT) \text{ and } +(LT) = -(TL),$$

so that (5) and (6) follow from (4).

291. In the fundamental equation

$$+(TM) = -(MT)$$

write M for T.

Then

$$+(MM) = -(MM).$$

Now the only algebraical quantity which is such that its addition is equivalent to its subtraction is *zero*.

Hence

$$(MM) = 0.$$

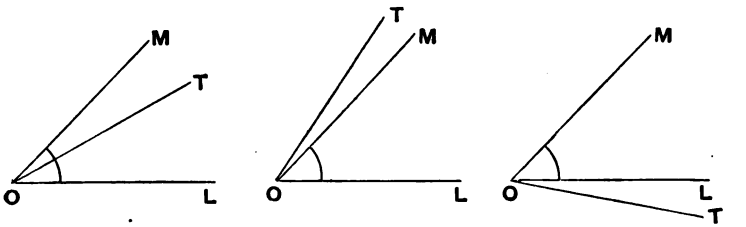
This, of course, only expresses the fact that the *distance from M to M* is nothing.

Directed Angles.

292. Similarly, if OL, OM make any angle LOM, and if OT be any other line through O in the plane LOM, the angle LOM will equal either

$$LOT + TOM \text{ or } LOT - MOT \text{ or } TOM - TOL,$$

according as OT is *between* OL and OM, or *beyond* OM, or *beyond* OL.



Using the symbol (LOM) to represent the angle LOM, regarded as described by the revolution of a line from OL to OM, and giving the same rules of interpretation for *directed angles* as for directed lengths, we may say universally

$$(LOM) = (LOT) + (TOM).$$

As before this equation is clearly realised by regarding the rotation from OL to OM as equivalent to a rotation from OL to OT followed by a rotation from OT to OM. But the equation can be *arithmetically* interpreted, only if OT is in the same [surface or] plane as the angle LOM.

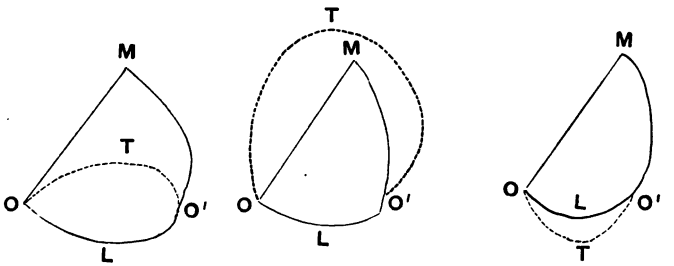
Directed Areas.

293. If OLO', $O'MO$ are two parts which together make up any closed line, and if OTO' be any other line joining OO', then the area of $OLO'M$ will equal either

$$OLO'T + OTO'M \text{ or } OLO'T - OMO'T \text{ or } OTO'M - OTO'L,$$

according as OTO' falls entirely *within* the figure $OLO'M$, or

outside the figure *beyond OMO'*, or *outside* the figure *beyond OLO'*.



Using the symbol (*OLO'MO*) to represent an area whose periphery is traced out by moving in the direction $O-L-O'-M-O$, and giving the same rules of interpretation for *directed areas* as for directed lengths and directed angles, we may write universally

$$(OLO'MO) = (OLO'TO) + (OTO'MO).$$

In tracing out the peripheries of the *two* figures whose sum is equal to the original figure, the tracing point moves *twice* over the part of their peripheries *which is common*, viz. *OTO'*;—first in the direction (*O'TO*), then in the direction (*OTO'*). These two movements, *being in opposite directions*, may be regarded as cancelling one another, and the resultant movement as equivalent to (*OLMO*).

Algebraical Representation of Directed Magnitudes.

294. In each of the three kinds of directed magnitudes above discussed,—viz. length, angle, area,—**two opposed** directions were involved.

Now in dealing with such magnitudes, we have to reduce them to the *same* direction before combining them arithmetically.

Hence it is convenient to choose *one* of the two directions in question as that to which *all* the directed magnitudes shall be reduced.

The direction so chosen is called the *Positive;* that opposite to it is called the *Negative.*

Again, the *addition* of any directed magnitude was interpreted to mean the same as the *subtraction* of the *oppositely* directed magnitude.

Hence, the algebraical signs of affection exactly answer the purpose of denoting two oppositely directed magnitudes.

It is clearly immaterial which direction we choose to represent by positive numbers and which by negative.

295. The method of applying *signs of affection* to denote directed angles is the same as that for directed lines.

In Chap. I., it is shown that a trigonometrical angle, corresponding to a given geometrical angle, may be of any *magnitude.*

When further we distinguish the *Initial* from the *Final* line, we can also apply *either sign* of affection to represent an angle.

In Arts. 14—16, the general value of the magnitude of a trigonometrical angle is given in terms of the corresponding geometrical angle. We may apply these results to a *directed* trigonometrical angle. For, *whichever direction the revolution takes,* a *complete* revolution brings the revolver back to its *initial* position, and a half-complete revolution brings it to the initial position *reversed.* Hence the following table gives the value of the trigonometrical directed angle corresponding to any position of the final line :—

Final line in	Directed angle =
1st Positive Quadrant	$2n \cdot 180° +$ a positive acute angle.
2nd " "	$(2n+1)\,180° -$ " " "
3rd " "	$(2n+1)\,180° +$ " " "
4th " "	$2n \cdot 180° -$ " " "

where n is 0 or any positive or negative integer.

Dependent Signs.

296. We may choose *arbitrarily* either direction of revolution as positive.

But it is convenient to make the sign for an *area* depend on that for a revolution. Thus

Let S be any point taken **inside** a closed area.

Let an elastic string tied to S revolve round S through four right-angles, and let its other extremity trace out the *periphery* of the area, so that the string just traverses the whole area.

Then we take the area as positive, when its periphery is traced out in the direction corresponding to the positive direction for the angle.

It should be noted that if S is taken *outside* the area, the string will have to revolve first through an angle in one direction and then through an equal angle in the opposite direction, in order that its extremity may trace out the periphery. Hence to connect the sign of an area with the sign of an angle we must take S *inside* the area.

297. When any line is taken independently, either direction along it may be *arbitrarily* chosen as *positive*.

But when it is essential to regard a certain line as *perpendicular* to another line (whose positive direction has already been assigned) the sign of the former depends on the latter.

Thus, the line, which makes a **positive** right-angle with an independently positive line, is *dependently* **positive**: and hence, the line, which makes a **negative** right-angle with an independently positive line, is *dependently* **negative**.

Signs in Multiplication.

298. Suppose $ABCD$ is any rectangle. (See fig. p. 9.)

Then, by an extension of the meaning of multiplication (explained in Art. 21) we have

$$\text{area } ABCD = \text{length } AB \times \text{length } BC.$$

By a still further extension we may say

directed area $(ABCDA) =$

$(AB) \times (BC)$ or $(BC) \times (CD)$ or $(CD) \times (DA)$ or $(DA) \times (AB)$,

directed area $(CBADC) =$

$(CB) \times (BA)$ or $(BA) \times (AD)$ or $(AD) \times (DC)$ or $(DC) \times (CB)$.

In each of these products the directed factors (and the letters composing those factors) are written in the order indicated by the designation of the directed area.

Let us choose arbitrarily the sign of the *first* factor. Then that of the second, since it is essential to regard it as perpendicular to the first, will be chosen dependently on the first.

Suppose, then, that $(ABCDA)$ is considered *positive*, and $(CBADC)$ is considered *negative*.

I. Let (AB) be arbitrarily chosen as *positive*, so that (CD) is *negative*.

Then (BC) makes a *positive* right-angle with AB, and is, therefore, dependently *positive*.

But (DA) makes a *negative* right-angle with AB, and is, therefore, dependently negative.

Thus, the rule of signs in multiplication are applicable to the four equations

$$(ABCDA) = (AB) \times (BC) = (CD) \times (DA),$$
$$(CBADC) = (BA) \times (AD) = (DC) \times (CB).$$

II. Let (BC) be arbitrarily chosen as *positive;* so that (DA) is *negative*.

Then (CD) makes a *positive* right-angle with (BC), and is, therefore, dependently *positive*.

But (AB) makes a *negative* right-angle with (BC) and is, therefore, dependently *negative*.

Thus the rule of signs in multiplication applies to the remaining four equations,

$$(ABCDA) = (BC) \times (CD) = (DA) \times (AB),$$
$$(CBADC) = (CB) \times (BA) = (AD) \times (DC).$$

299. The student must carefully distinguish between the multiplication of a directed length *by a directed length* and the multiplication of a directed length by a *number, positive or negative.*

$$\text{Length} \times \textit{Length} = \textit{Area}.$$
$$\text{But Length} \times \textit{Number} = \textit{Length}.$$

Thus, (3 ft.) × (− 2 ft.) may be interpreted as − 6 square ft.

But (3 ft.) × (− 2) must be interpreted as − 6 ft.

Hence multiplying a directed length by a *negative* number has the effect of altering the length and *reversing its direction.*

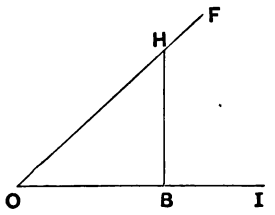
§ 2. Application to Trigonometrical Ratios.

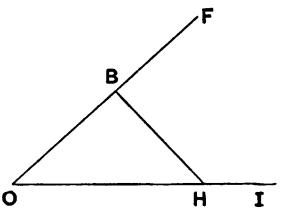
300. Let a straight line, terminated at a fixed apex, describe an angle by revolving in a plane from an initial to a final position.

Let a transverse line, perpendicular to either the initial or the final line, cut both these lines (either being produced if necessary). Then

(1) The part cut off from the transversal—*directed from the initial to the final line*—is called the **Perpendicular.**

(2) The parts cut off from the initial and final lines—*directed from the apex of the angle*—are called **Base** and **Hypothenuse** (that being hypothenuse which is opposite the right-angle.)





Thus, let *OI* be the *initial*, *OF* the *final* line.

Then if *B* is on *OI*, and *BH* perpendicular to *OI*;
(*OH*) is hypothenuse; (*OB*) is base; and (*BH*) is perpendicular.

But if B is on OF, and BH perpendicular to OF;

(OH) is hypothenuse; (OB) is base; and (HB) is perpendicular.

Having then defined base, hypothenuse, and perpendicular as *directed* lengths, the ratios of *any directed angle whatever* are defined in the same way as those of an *acute* angle were defined in Art. 71—73, *except that directed lengths are substituted for mere lengths.*

Conventions as to Signs.

301. In order to express the Trigonometrical Ratios *algebraically* we must affix *signs of affection* to the four directed magnitudes in question, viz.:

(1) The angle. (2) The initial line. (3) The final line. (4) The transverse line.

Mathematicians adopt the following conventions in order to express algebraically the ratios of any directed angle:—

(1) Lengths along either bounding line of the angle are regarded as *positive.*

(2) Lengths *perpendicular* to either bounding line have the same sign as the right-angle which they make with that bounding line.

Thus, the bounding lines being assumed to be positive, the perpendiculars to them receive a *dependent* sign, as explained in Art. 297.

302. The mathematician thus leaves two directions to be arbitrarily chosen, viz.:—

(1) The *positive* direction of revolution.

(2) The direction of the *initial* line.

303. It is very desirable to fix upon some constant directions of these two, for the purpose of *mentally retaining* the signs of the different ratios of different angles.

For this purpose we have what may be called a *teacher-and student's convention*; viz.:

(1) The *positive* direction of revolution shall be *opposite* to the motion of the hands of a clock; i.e. a *Right-Up-Left-Down-*

Right motion; so that the *negative* direction is the *same* as that of the motion of the hands of a clock; i.e. a *Left-Up-Right-Down-Left* motion.

(2) The initial line shall be supposed to be drawn *right-wards.*

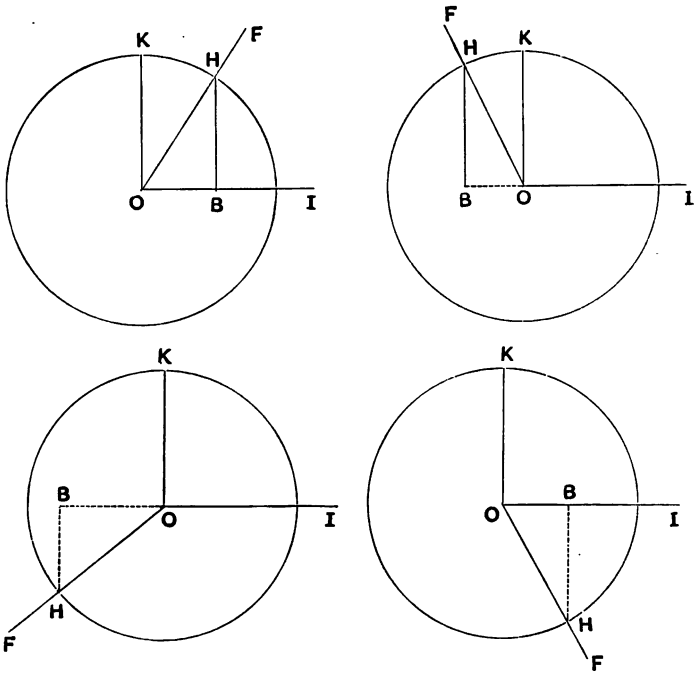
Since a line drawn *upwards* makes (in accordance with this convention) a *positive* right-angle with a line drawn *right-wards*, we have, combining the Mathematician's with the Teacher-and-Student's convention, the following three positive directions:

(1) For initial line, *Right-wards.*

(2) For revolution, *Right-upwards.*

(3) For perpendicular, *Upwards.*

304. *To determine the signs of the ratios for the different quadrants in which the final line may lie.*



Let OI be the initial line, OF the final line.

For simplicity, take the hypothenuse (OH) in every case along OF. So that **(OH) is always positive.**

Draw HB perpendicular to OI or OI produced.

Then (OB) is the base; and (BH) is the perpendicular.

Let (OK) make a *positive* right-angle with (OI), so that lines in the direction (OK) are *positive*.

(1) When OF is in the 1st quadrant,

(BH) which is drawn in the direction (OK) is **Positive.**

(OB) „ „ in the direction (OI) is **Positive.**

(2) When OF is in the 2nd quadrant,

(BH) „ „ in the direction (OK) is **Positive.**

(OB) „ „ oppositely to (OI) is **Negative.**

(3) When OF is in the 3rd quadrant,

(BH) „ „ oppositely to (OK) is **Negative.**

(OB) „ „ oppositely to (OI) is **Negative.**

(4) When OF is in the 4th quadrant,

(BH) „ „ oppositely to (OK) is **Negative.**

(OB) „ „ in the direction (OI) is **Positive.**

Thus the signs corresponding to the four quadrants are

for Perp.; (1) +; (2) +; (3) −; (4) −.

for Base; (1) +; (2) −; (3) −; (4) +.

Now since we have taken the hypothenuse (OH) to be *positive* in every case,

(I) **Sine** and **Cosecant** have the sign of the **Perp.**

(II) **Secant** and **Cosine** have the sign of the **Base.**

Also (III) **Tangent** and **Cotangent** have the sign obtained by *division* of **Base** and **Perp.**

The following table gives all the results:

Quadrants	1st	2nd	3rd	4th
Sine and Cosecant	+	+	−	−
Tangent and Cotangent	+	−	+	−
Secant and Cosine	+	−	−	+

Here observe that, while *every* ratio is positive in the 1st quadrant,

the *sine* (and cosecant) are also positive in the 2nd;
the *tangent* (and cotangent) „ „ in the 3rd;
the *secant* (and cosine) „ „ in the 4th.

305. In the above figures,

(1) The initial line OI is drawn *Rightwards:*

(2) The positive direction of revolution is *Right-upwards:* so that

(3) OK, which is drawn *Upwards*, is positive.

Also the lines (OB), (BH) are *dotted* when *negative.*

Retaining these figures in the mind, it is easy to remember that

(OB) is $+^{ve}$, when *Rightwards*; $-^{ve}$, when *Leftwards.*
(BH) is $+^{ve}$, when *Upwards*; $-^{ve}$, when *Downwards.*

306. But the student must particularly observe, that the signs here ascribed to the base and perpendicular, according as the former is rightwards or leftwards and the latter upwards or downwards, apply only to the *angle* IOF in each of the four figures.

Thus, in each figure, when we regard (OB) as the base of the acute angle BOH, it is *positive:* so that (e.g.) cos (BOH) is *positive* in each case.

Moreover, retaining the same positive direction of revolution viz. *Right-Up-Left-Downwards*,

in 2nd figure, sin (BOH) is *negative*.

in 3rd figure, sin (BOH) is *positive*.

Comparison of the ratios of related angles.

307. In denoting the ratios of angles, we may use either symbols for *directed* lengths, or symbols for *mere Arithmetic* lengths —*prefixing explicitly the proper signs*.

The latter method was used in Art. 136, in defining the ratios of obtuse angles. The student ought to be able to apply either method when required.

308. In each of the figures of Art. 304, let $\angle BOH$ represent the *same positive acute angle* A.

Then the ratios of $\angle BOH$ are all positive.

Now, in the *first* figure, $\angle IOF$ represents the positive angle A, or generally the angle $2n \cdot 180° + A$.

In the *second* figure, $\angle IOF$ represents the positive angle $180° - A$, or generally the angle $(2n+1)\,180° - A$.

In the *third* figure, $\angle IOF$ represents the positive angle $180° + A$, or generally the angle $(2n+1)\,180° + A$.

In the *fourth* figure, $\angle IOF$ represents the negative angle $-A$, or generally the angle $2n \cdot 180° - A$.

Thus, from *first* figure,

any ratio of $2n \cdot 180° + A$ = that same ratio of A.

From *second* figure,

$$\sin(180° - A) = \frac{BH}{OH} = \sin A.$$

$$\cos(180° - A) = \frac{-OB}{OH} = -\cos A.$$

$$\tan(180° - A) = \frac{BH}{-OB} = -\tan A.$$

From *third* figure,

$$\sin(180° + A) = \frac{-BH}{OH} = -\sin A.$$

$$\cos(180° + A) = \frac{-OB}{OH} = -\cos A.$$

$$\tan(180° + A) = \frac{-BH}{-OB} = \tan A.$$

From *fourth* figure,

$$\sin(-A) = \frac{-BH}{OH} = -\sin A.$$

$$\cos(-A) = \frac{OB}{OH} = \cos A.$$

$$\tan(-A) = \frac{-BH}{OB} = -\tan A.$$

Thus any ratio has the same *numerical* value for all the angles found by adding or subtracting A from O or 180° or any multiple of 180°.

But the *sign* of the ratio has to be found by considering in what *quadrant* the final line would lie for the angle in question.

309. The above results may be expressed as follows:—

If A is an acute angle, and n any integer.

I. Any ratio of $2n \,.\, 180° + A$ = that same ratio of A.

II. The sine or cosecant of $(2n + 1)\, 180° - A$ is the *same* as that of A; but its other ratios are the *negatives* of those of A.

III. The tangent or cotangent of $(2n + 1)\, 180° + A$ is the *same* as that of A; but its other ratios are the *negatives* of those of A.

IV. The secant or cosine of $2n \,.\, 180° - A$ is the *same* as that of A; but its other ratios are the *negatives* of those of A.

310. The above results hold *whatever value A may have.*

This may be shown in the following way.

The revolution $2n \,.\, 180°$ carries the revolver from OI back again to OI.

The revolution $(2n+1)\,180°$ carries the revolver from OI to OI' (i.e. OI reversed).

The following table shows into what quadrant the revolver is carried by a revolution equal to $+A$ or $-A$ from OI or from OI' according as the revolution A from OI carries the revolver into the 1st, 2nd, 3rd, or 4th Quadrant.

Revolution	Quadrants			
$+A$ from OI	1	2	3	4
$-A$ from OI'	2	1	4	3
$+A$ from OI'	3	4	1	2
$-A$ from OI	4	3	2	1

Now the *sine* and *cosecant* have the same sign in the first as in the second quadrant; and also in the third as in the fourth, Hence the above table shows that

angles in the series $(2n+1)\,180° - A$

have the same *sine* and *cosecant* as A.

Again the *tangent* and *cotangent* have the same sign in the first as in the third quadrant, and also in the second as in the fourth. Hence the above table shows that

angles in the series $(2n+1)\,180° + A$

have the same *tangent* and *cotangent* as A.

Lastly the *secant* and *cosine* have the same sign in the first as in the fourth quadrant, and also in the second as in the third. Hence the above table shows that

angles in the series $2n.\,180° - A$

have the same *secant* and *cosine* as A.

311. The results of the last article may be expressed in other forms, differing only verbally from the above.

A.

I. Angles in the series $2n \cdot 180° + A$ have *every* ratio the same.

II. Angles in the series

$$2n \cdot 180° + A \text{ and } (2n+1)\,180° - A,$$

i.e. in the series, $m \cdot 180° + (-1)^m A$

are equi-sinal and equi-cosecantal.

III. Angles in the series

$$2n \cdot 180° + A \text{ and } (2n+1)\,180° + A,$$

i.e. in the series, $m \cdot 180° + A$

are equi-tangential and equi-cotangential.

IV. Angles in the series

$$2n \cdot 180° + A \text{ and } 2n \cdot 180° - A,$$

i.e. in the series, $2m \cdot 180° \pm A$

are equi-cosinal and equi-secantal.

B.

I. Angles whose difference is a multiple of 360° have *all* their ratios the same.

II. Angles whose difference is an even or whose sum is an odd multiple of 180° are *equi-sinal.*

III. Angles whose difference is *any* multiple of 180° are *equi-tangential.*

IV. Angles whose sum or difference is a multiple of 360° are *equi-cosinal.*

312. The *converses* of the above propositions are also true:

I. Any angle which has *two non-reciprocal* ratios the same as A must belong to the series

$$2m \cdot 180° + A.$$

II. Any angle which has the same *sine* or *cosecant* as A must belong to the series

$$m \cdot 180° + (-1)^m A.$$

III. Any angle which has the same *tangent* or *cotangent* as A must belong to the series

$$m \,.\, 180° + A.$$

IV. Any angle which has the same *secant* or *cosine* as A must belong to the series

$$2m \,.\, 180° \pm A.$$

For only one positive acute angle can have any ratio of a given value (Art. 68).

And it is by an *acute* angle that the numerical values of the ratios of *any* angle are determined.

313. Hence the *general* solutions of trigonometrical equations are as follows, where A is *any one* solution.

I. Given $\sin^2\theta = a$, or $\cos^2\theta = a$, &c.
then $\theta = n \,.\, 180° \pm A.$

II. Given $\sin\theta = a$,
then $\theta = n \,.\, 180° + (-1)^n A.$

III. Given $\tan\theta = a$,
then $\theta = n \,.\, 180° + A.$

IV. Given $\cos\theta = a$,
then $\theta = n \,.\, 360° \pm A.$

Example. Give the general solution of the equation

$$\sin\theta + 2\cos^2\theta = 2.$$

This equation may be written

$$\sin\theta = 2(1 - \cos^2\theta) = 2\sin^2\theta,$$
$$\therefore \sin\theta - 2\sin^2\theta = 0,$$
$$\therefore \sin\theta(1 - 2\sin\theta) = 0,$$
$$\therefore \text{either } \sin\theta = 0, \quad \text{or } (1 - 2\sin\theta) = 0.$$

Now if $\sin\theta = 0$, $\theta = 0$ or $180°$ or $360°$, &c.

$\therefore$ the general solution of $\sin\theta = 0$ is $\theta = n \,.\, 180°$.

But if $1 - 2\sin\theta = 0$, then $\sin\theta = \frac{1}{2}$;

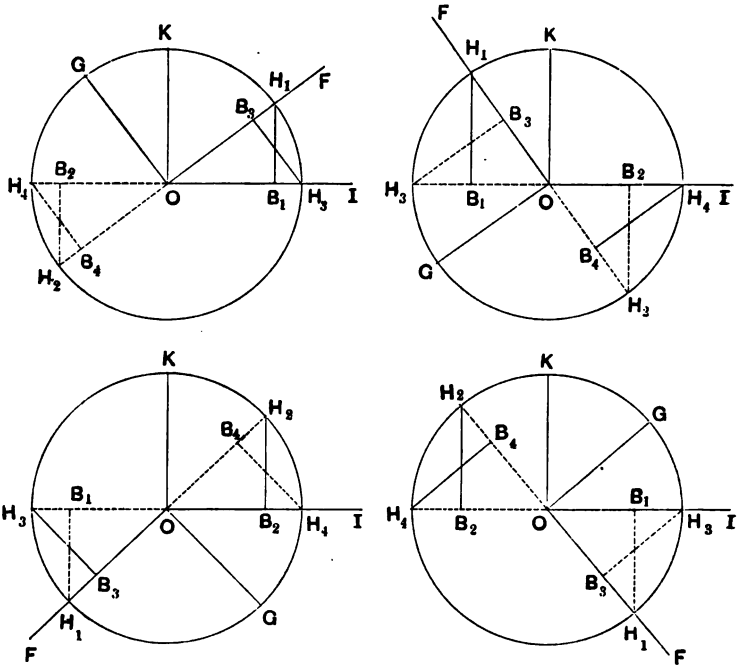
$\therefore$ $\theta = 60°$ or $120°$ or $420°$, &c.;

i.e. $\theta = n \,.\, 180° + (-1)^n 60°$,

$\therefore$ the general solution of the given equation is

$$\theta = n \,.\, 180° \text{ or } n \,.\, 180° + (-1)^n 60°.$$

314. It will be useful to show that the fundamental results (as to the signs of the ratios corresponding to the four quadrants) given in Art. 304, follow however the right-angled triangle may be formed.



There are four ways of forming the triangle in each case.

I. Hypothenuse positive and in final line (perp. B_1H_1).

II. Hypothenuse negative and in final line (perp. B_2H_2).

III. Base positive and in final line (perp. H_3B_3).

IV. Base negative and in final line (perp. H_4B_4).

Let OK make a positive right-angle with OI,

and OG make a positive right-angle with OF.

Then

When OF is in *first* quadrant,

B_1H_1 like OK is +; OB_1 like OI is +; OH_1 like OF is +.

B_2H_2 unlike OK is −; OB_2 unlike OI is −; OH_2 unlike OF is −.

H_3B_3 like OG is +; OB_3 like OF is +; OH_3 like OI is +.

H_4B_4 unlike OG is −; OB_4 unlike OF is −; OH_4 unlike OI is −.

It follows that the ratios of the angle IOF are *all* positive however the right-angled triangle be constructed.

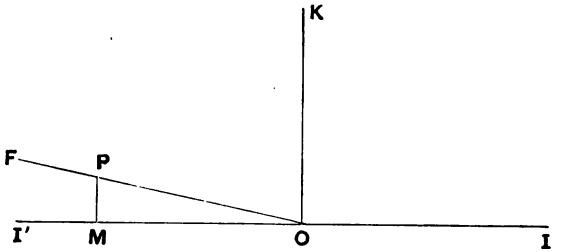
The student should in the same way examine the case for the 2nd, 3rd, and 4th quadrants.

Limiting values of the ratios.

315. In Art. 113, we have shown that

for angle 0°; perp. = 0 and base = hyp.

for angle 90°; perp. = hyp. and base = 0.



Consider angle 180°.

Let the angle IOF be nearly 180°; then MP, the perpendicular, is very small; and OM, the base, is *in magnitude very nearly equal to* OP.

But, since (OM) is directed *opposite* to OI, the *initial* line, we have, (representing OM algebraically),

(OM) = − arithmetic distance between O and M.

But since (OP) is directed *the same way* as OF, the *final* line, we have, (representing OP algebraically),

$$(OP) = +\text{arithmetic distance between } O \text{ and } P.$$

Now at the limit, (MP) vanishes, and (OM) coincides with (OP). Thus

$$\sin 180° = 0 \text{ and } \cos 180° = -1.$$

This last result is deserving of special attention.

In representing (OM) by an algebraical quantity we refer to (OI) in order to determine its sign: but in representing (OP) we refer to (OF).

In the limit (OF) becomes opposite to (OI); and hence the signs of (OM) and (OP) are *opposite;* although they exactly coincide in direction as well as magnitude.

This exemplifies the principle that lines which coincide (so to speak) *purely incidentally* have not necessarily the same sign.

Consider the angle 270°.

Here (OM) vanishes, and (MP) ultimately coincides with (OP).

But (MP) makes a *negative* right-angle with the *initial* line (OI), and is therefore represented algebraically by a *negative* number.

On the other hand, (OP) which is *along* the *final* line is represented by a *positive* number.

Hence

$$\sin 270° = -1 \text{ and } \cos 270° = 0.$$

316. The principle of continuity confirms these conclusions.

For, when the final line is in the 2nd or 3rd quadrant, the *cosine* is negative. Hence, when it separates these quadrants, viz. when the angle is $(2n+1)\,180°$, its cosine must be negative.

Again, when the final line is in the 3rd or 4th quadrant, the *sine* is negative. Hence, when it separates these quadrants, viz. when the angle is $(4n-1)\,90°$, its sine must be negative.

317. *To trace the changes in magnitude and sign of the ratios of an angle as it increases from* 0 *to* 360°.

As an example, take the *tangent.*

From 0° to 90°, tangent is $+^{ve}$, and increases from 0 to $+\infty$.
From 90° to 180°, tangent is $-^{ve}$, and increases from $-\infty$ to 0.
From 180° to 270°, tangent is $+^{ve}$, and increases from 0 to $+\infty$.
From 270° to 360°, tangent is $-^{ve}$, and increases from $-\infty$ to 0.

It should be observed that the ratios which become *infinite* at the limiting values 0°, 90°, 180°, 270° are *ambiguous* in sign: i.e. their sign depends upon whether we arrive at the limit by *increasing* the angle *up* to its limit or by *decreasing* the angle *down* to its limit.

318. *To examine how the fundamental formulæ connecting the ratios of an angle are affected by taking the angle of any magnitude or sign.*

The formulæ of Art. 79, which follow immediately from definition, are evidently unaffected by extending the angle to any magnitude or sign.

Also the formulæ of Art. 80, which involve the *squares* of the ratios, are unaffected by the *signs* of the ratios.

But, in applying these latter to express a ratio in term of another by taking the square root, we must observe that

$$\sin A = \pm\sqrt{(1-\cos^2 A)};\ \operatorname{cox} A = \pm\sqrt{(1+\cot^2 A)};$$
$$\tan A = \pm\sqrt{(\sec^2 A - 1)};\ \cot A = \pm\sqrt{(\operatorname{cox}^2 A - 1)};$$
$$\sec A = \pm\sqrt{(1+\tan^2 A)};\ \cos A = \pm\sqrt{(1-\sin^2 A)}.$$

The upper sign must be taken
in the 1st line, for angles of the 1st or 2nd quadrant;
in the 2nd line, for angles of the 1st or 3rd quadrant;
in the 3rd line, for angles of the 1st or 4th quadrant.

319. This result shows that, knowing one ratio of an angle and *nothing more*, there are two possible equal and opposite values for any other ratio (except the reciprocal of that given).

This is geometrically obvious.

For example; knowing that the *cotangent* of an angle has some particular value, which is (say) *negative*, the angle may be in the 2nd or 4th quadrant; and, therefore, either sine, cosine, secant, or cosecant may be *positive* or *negative*.

§ 3. Inverse Symbols.

320. The symbols $\sin^{-1}$, $\cos^{-1}$ &c. are used in the following sense.

If θ be any angle, whose sine has some given value a, then for θ we may write $\sin^{-1} a$.

And so for the other ratios.

Thus the equations

$$\sin\theta = a \text{ and } \theta = \sin^{-1} a$$

are precisely equivalent statements.

The latter is obtained from the former by symbolically '*dividing*' both sides by the function '*sin*', as if *sin* were an algebraic factor.

In working out problems involving these (so called) **inverse** symbols, it is only necessary to equate each inverse symbol to some new letter, and translate the resulting equation into the ordinary form by means of the equivalence above explained.

This method is similar to that required in logarithms.

Example. To show that

$$\tan^{-1} a + \tan^{-1} b = \tan^{-1} \frac{a+b}{1-ab}.$$

Let $\tan^{-1} a = \theta$, *this means* $a = \tan\theta$.

Let $\tan^{-1} b = \phi$, *this means* $b = \tan\phi$.

Now
$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

$$\text{i.e. } \frac{a+b}{1-ab}.$$

Translating this last equation back into the inverse notation,

$$\theta + \phi, \text{ i.e. } \tan^{-1} a + \tan^{-1} b, = \tan^{-1} \frac{a+b}{1+ab}.$$

321. The following points must be carefully observed :—

(1) Whereas $\sin\theta$ is a *numerical quantity* $= a$ (say); $\sin^{-1} a$ is an *angle*.

(2) Whereas, when θ is given, $\sin\theta$ has only *one* value $= a$ (say); when a is given, $\sin^{-1} a$ has an *indefinite number* of values.

In fact the results of Art. 313, expressed in inverse notation, become as follows :—

General value of $\sin^{-1} a = n \,.\, 180° + (-1)^n$ [some particular value of $\sin^{-1} a$]

General value of $\tan^{-1} a = n \,.\, 180° +$ [some particular value of $\tan^{-1} a$]

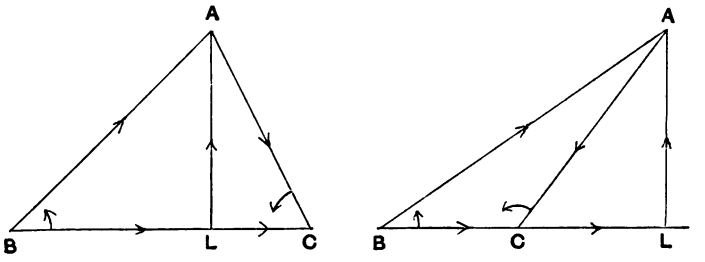
General value of $\cos^{-1} a = n \,.\, 360° \pm$ [some particular value of $\cos^{-1} a$]

where n has any integral value whatever.

§ 4. Adaptation of Results to a Triangle.

322. In applying the theory of directed magnitudes to the formulæ of a triangle, it must be noticed that we have *three* primary lines to consider.

Take the triangle ABC.



Consider the results obtained by dropping AL perpendicular upon BC. We have

$$\cos(CBA) = \frac{(BL)}{(BA)} \text{ and } \sin(CBA) = \frac{(LA)}{(BA)}.$$

$$\therefore (BA)\cos(CBA) = (BL) \text{ and } (BA)\sin(CBA) = (LA).$$

Also

$$\cos(ACB) = \frac{(CL)}{(CA)} = \frac{(LC)}{(AC)} \text{ and } \sin(ACB) = \frac{(AL)}{(CA)} = \frac{(LA)}{(AC)}.$$

$$\therefore (AC)\cos(ACB) = (LC) \text{ and } (AC)\sin(ACB) = (LA).$$

Comparing these results, since $(BC) = (BL) + (LC)$,

$$(BA)\cos(CBA) + (AC)\cos(ACB) = (BC),$$

$$(BA)\sin(CBA) = (AC)\sin(ACB).$$

Also the triangle $(ABCA) = \frac{1}{2}(BC)(LA)$.

323. Now let us take the area $(ABCA)$ to be *positive*; so as to correspond to the *positive* rotation from (BC) to (BA), i.e., the angle (CBA); and to that from (AC) produced to (BC) produced, i.e. the angle (ACB).

Then take (BC) as an initial line, which will, therefore, be *primarily* positive.

Hence (LA), which makes a positive right-angle with (BC), is *secondarily* positive.

Of the angle (CBA), (BC) is the initial, (BA) is the final line; these are, therefore, *positive.*

Of the angle (ACB), (CA) is the initial line *reversed*, (CB) is the final line *reversed;* these are, therefore, *negative.*

(LC) is positive or negative according as it is like or unlike (BC); i.e. according as C is acute or obtuse.

(BL) is positive or negative according as it is like or unlike (BC); i.e. according as B is acute or obtuse.

The signs in all the above equations are, therefore, accounted for.

324. We have, therefore, *three* separate sets of conventions according to the side and its perpendicular which we consider.

In each case, however, we maintain the direction $(ABCA)$ or $(BCAB)$ or $(CABC)$ as *positive* for the area.

Then, in the three cases, we take (BC), (CA), (AB) as *primarily* positive.

And hence (LA), (MB), (NC)—which make *positive* right-angles with these—are *secondarily* positive.

But the three sets of conventions must not otherwise be combined.

For, as we see in Art. 322, when we drop a perpendicular on (BC), (BA) and (AC) (which follow the order B, C) are regarded as positive: whereas they are each negative, when taken by themselves as initial lines.

325. The convention, at which we have arrived with respect to perpendiculars on the sides, is of great importance. It is as follows:

Perpendiculars drawn *from* a side *towards* the opposite angle are *positive*.

Example 1. If J be any point, and if x, y, z represent perpendiculars from J to the sides—directed *from* the sides *to* J—and a, b, c represent the lines (BC), (CA), (AB), and if Δ represent the area $(ABCA)$; then in every case

$$ax+by+cz=2\Delta.$$

This follows from Ex. x. 110.

Example 2. Let O be the orthocentre of ABC; and AL perpendicular upon BC. Apply the convention of signs to the algebraical expressions for LO and OA.

If ABC is an acute-angled triangle we see from Art. 212, that $LO=2R\cos B\cos C$; and $OA=2R\cos A$.

Now, if A is *obtuse*, OA must be drawn *towards* BC; because O is within the space between BA and CA *produced*. But in this case $\cos A$ is negative. Hence we have always

$$(OA)\ directed=2R\cos A.$$

Again, if B is obtuse, (LO) must be drawn *away from* A, because O is within the space between CB and AB produced. But in this case $\cos B$ is negative. Hence we have always

$$(LO)\ directed=2R\cos B\cos C.$$

Furthermore, $$(LO)+(OA)=(LA).$$

Now $$\begin{aligned}2R\cos B\cos C+2R\cos A&=2R\cos B\cos C-2R\cos(B+C)\\&=2R\sin B\sin C,\\&=b\sin C,\end{aligned}$$

which is the obvious value of (LA).

Examples XII.

1. If A, B, C, D, E are points taken in order on a line, find the directed length which is equivalent to each of the following combinations of directed lengths:—

(1) $(AB)+(BC)+(CD)+(DE)$. (2) $(AC)+(CE)-(DE)$.

(3) $(BE)+(ED)-(CD)$. (4) $(DE)+(EA)-(DC)$.

(5) $(CA)+(AE)-(DE)+(DB)$. (6) $(AD)+(DB)+(BE)+(EC)+(CA)$.

2. Show how symbols denoting directed lengths may be used in solving the following problem:—

Two persons A and B walk along a road in the same direction:—A, at the rate of a miles an hour; B, at the rate of b miles an hour. At a certain moment A is at a place c miles from a town O (towards which he is then walking), and B at a place d miles from the same town on the same side of it. Find where and when A and B are to be found together: distinguishing the cases according as $a > \text{or} < b$, $c > \text{or} < d$, $a/c > \text{or} < b/d$.

Prove the following identities:—(3—8).

3. $(\sin 150° - \cos 150°)(\sin 120° + \cos 120°) = \cos(-60°)$.

4. $(\sin 225° + \sin 240°)(\cos 210° - \cos 225°) = \sin^2 150°$.

5. $\sin 210° + \sin 240° = \sqrt{2}\cos 165°$.

6. $\cos(-30°) + \cos 120° = \sqrt{2}\sin 165°$.

7. $\cos(-36°) = \cos 300° + \cos 288°$.

8. $\sec 3\pi = \tan 855° = \operatorname{cosec} \frac{1}{2}7\pi = 2\cos 840°$.

9. Given $\sec A = 7$, find $\tan A$ and $\operatorname{cosec} A$. Explain the reason why each result is ambiguous.

10. Given $180° > A > 90°$, and $\sin A = \frac{1}{3}$; find $\cos A$.

11. Given $270° < A < 360°$, and $\tan^2 A = \frac{1}{4}$, find $\sin A$.

12. Give the complete solutions of the following equations:—(13—32).

13. $\sin A = \frac{1}{2}$. 14. $\tan A = \dfrac{1}{\sqrt{3}}$. 15. $\cos A = \dfrac{\sqrt{3}}{2}$.

16. $\tan A = \tan 19°$. 17. $2\sin^2 A = 1$. 18. $\sqrt{2}\,.\sin A = 1$.

19. $\sin^2\theta = \cos^2\theta$. 20. $\sin\theta = \cos\theta$. 21. $\tan^2\theta = 3$.

22. $2\cos x + 2\sec x = 5$. 23. $3\tan^2\theta + 8\cos^2\theta = 7$.

24. $8\cos^4\theta + 10\sin^2\theta = 7$. 25. $\operatorname{cosec}\theta = 2\sin\theta$.

26. $2\sin A = \tan A$. 27. $4 + 4\cos A = 3\sec A$.

28. $\cot\theta - \tan\theta = \cos\theta + \sin\theta$. 29. $\sec^3\theta - 2\tan^2\theta = 2$.

30. $\tan p\theta = \cot q\theta$. 31. $\operatorname{vers}\theta = 0$. 32. $\tan\theta = \pm 1$.

33. Find the values of the ratios of the following angles:—

$$150°, \frac{4\pi}{3}, 250^g, 765°, -60°, -\frac{2\pi}{3}, 945°, 108°, -\frac{\pi}{8}, 2n\pi + \frac{\pi}{4},$$

$$(2n+1)\pi - \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}.$$

34. Trace the changes in magnitude and sign of each of the ratios of an angle which increases from 0 to 360°.

35. Show that a ratio always changes its sign, as it passes through the value 0 or ∞.

36. Show that for *any* value of A, $\sin(90° - A) = \cos A$; $\cos(90° - A) = \sin A$; $\sin(90° + A) = \cos A$; $\cos(90° + A) = -\sin A$.

37. Show that

$$\sin\left\{(4n+1)\frac{\pi}{2} - A\right\} = \cos A = \sin\left\{(4n+1)\frac{\pi}{2} + A\right\},$$

and that

$$\sin\left\{(4n+3)\frac{\pi}{2} - A\right\} = -\cos A = \sin\left\{(4n+3)\frac{\pi}{2} + A\right\}.$$

38. Trace the changes in sign and magnitude as A increases from 0 to 360° of

$\sin A + \cos A$; $\sin A - \cos A$; $\tan A + \cot A$; $\tan A - \cot A$;

$\sin A\,.\cos A$; $\tan A\,.\sec A$; $\cos A\,.\cot A$; $\dfrac{\cos A + \sin A}{\cos A - \sin A}$.

[The values of these expressions at the beginning and end of each *octant* should be given.]

39. Prove

(i) If n is even, $\sin n\left(\frac{1}{2}\pi - \theta\right) = (-1)^{\frac{n}{2}-1} \sin n\theta$;

(ii) If n is even, $\cos n\left(\frac{1}{2}\pi - \theta\right) = (-1)^{\frac{n}{2}} \cos n\theta$;

(iii) If n is odd, $\sin n\left(\frac{1}{2}\pi - \theta\right) = (-1)^{\frac{n-1}{2}} \cos n\theta$;

(iv) If n is odd, $\cos n\left(\frac{1}{2}\pi - \theta\right) = (-1)^{\frac{n-1}{2}} \sin n\theta$.

40. Show that $\frac{1}{2}(4n+1)\pi \pm \left(\frac{1}{2}\pi - A\right)$ and $n\pi + (-1)^n A$ represent the same series of angles.

41. Show that

(i) $\sin^{-1} a \pm \cos^{-1} \sqrt{(1-a^2)} = n\pi$,

(ii) $\sec^{-1} a \pm \tan^{-1} \sqrt{(a^2-1)} = n\pi$,

(iii) $\cot^{-1} a \pm \operatorname{cosec}^{-1} \sqrt{(a^2+1)} = n\pi$,

if one *or* other of the signs is taken in each case.

42. If we interpret $\sqrt{m}$ to mean the *positive* quantity whose square is m; and $\sin^{-1} a$ to mean the numerically smallest angle, whose sign is the same as that of a, and whose sine $= a$; and so $\cos^{-1} a$, $\tan^{-1} a$ &c.: show that

(i) $\sqrt{a^2} \cdot \sin^{-1} a = a \cdot \cos^{-1} \sqrt{(1-a^2)}$,

(ii) $\sqrt{a^2} \cdot \tan^{-1} a = a \cdot \sec^{-1} \sqrt{(a^2+1)}$,

(iii) $\sqrt{a^2} \cdot \cos^{-1} a = \sqrt{a^2} \cdot \sin^{-1} \sqrt{(1-a^2)} + \frac{1}{2}\pi\left(a - \sqrt{a^2}\right)$,

(iv) $\sqrt{a^2} \cdot \sec^{-1} a = \sqrt{a^2} \cdot \tan^{-1} \sqrt{(a^2-1)} + \frac{1}{2}\pi\left(a - \sqrt{a^2}\right)$,

(v) $\sqrt{a^2} \cdot \sin^{-1} a + a \cdot \cos^{-1} a = \frac{1}{2}\pi \cdot \sqrt{a^2}$.

CHAPTER XIII.

RATIOS OF COMPOUND ANGLES.

§ 1. Sum and Difference of two Angles.

326. *To find the sine and cosine of* $A + 90°$ *and of* $A - 90°$, *for all values of* A, *in terms of the ratios of* A.

Let OI be an initial, and OF a final line.

Let OK make a positive right-angle with OI, so that angle IOK *is a positive right-angle.*

Let IO be produced to I', so that KOI' *is a positive right-angle.*

Then in determining the ratios of IOF,

OI is the direction for positive **bases**,
OK is the direction for positive **perps.**

But in determining the ratios of KOF,

OK is the direction for positive **bases**,
OI' is the direction for positive **perps.**

Hence, for any given position of OF,

base for IOF = − perp. for KOF,
perp. for IOF = + base for KOF.

$\therefore \cos IOF = -\sin KOF$ .....(1),

and $\sin IOF = +\cos KOF$ .....(2).

I. Then let $KOF = A$, then $IOF = A + 90°$.

$\therefore$ (1) becomes $\cos(A + 90°) = -\sin A$ .....(3),

(2) becomes $\sin(A + 90°) = +\cos A$ .....(4).

II. Let $IOF = A$, then $KOF = A - 90°$.

$\therefore$ (1) becomes $\sin(A - 90°) = -\cos A$ .....(5),

(2) becomes $\cos(A - 90°) = +\sin A$.....(6).

327. These formulæ may be applied to find the ratios of $A \pm 180°$, $A \pm 270°$ &c., in terms of ratios of A. Thus

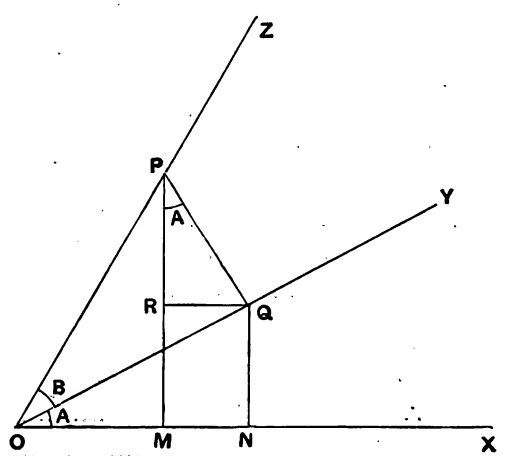
$$\sin(A + 180°) \text{ i.e. } \sin\{(A + 90°) + 90°\} = \cos(A + 90°) \text{ by (4)}$$
$$= -\sin A \text{ by (3)},$$

$$\cos(A + 180°) \text{ i.e. } \cos\{(A + 90°) + 90°\} = -\sin(A + 90°) \text{ by (3)}$$
$$= -\cos A \text{ by (4)},$$

and so on.

328. In the next articles we give the usual proofs of the formulæ for the sine and cosine of the sum and difference of two angles, *confining ourselves to positive acute angles.*

329. *To find the sine, cosine, and tangent of the* sum of two angles, *in terms of the ratios of the angles.*



Let $XOY = A$, and $YOZ = B$, then $XOZ = A + B$.

In OZ, *the final line of the compound angle* $A + B$, take any point P.

From P, draw PM, PQ perpendiculars upon OX, OY—*the initial and final lines of* A—respectively.

From Q, draw QN, QR perpendiculars upon OX, MP respectively.

Then $\angle QPR = 90° - \angle RQP = \angle OQR = \angle XOQ = A$.

[Or: RP, QP being perpendiculars upon OX, OY contain the same angle as OX, OY.]

(1) $\sin(A+B) = \sin XOZ = \dfrac{MP}{OP} = \dfrac{MR+RP}{OP} = \dfrac{NQ}{OP} + \dfrac{RP}{OP}$

= [introducing required hyps. under the numerators]

$$\frac{NQ}{OQ} \cdot \frac{OQ}{OP} + \frac{RP}{QP} \cdot \frac{QP}{OP}$$

$= \sin NOQ \,.\, \cos QOP + \cos QPR \,.\, \sin QOP$

$= \sin A \,.\, \cos B + \cos A \,.\, \sin B.$

(2) $\cos(A+B) = \cos XOZ = \dfrac{OM}{OP} = \dfrac{ON-MN}{OP} = \dfrac{ON}{OP} - \dfrac{RQ}{OP}$

= [introducing required hyps. under the numerators]

$$\frac{ON}{OQ} \cdot \frac{OQ}{OP} - \frac{RQ}{QP} \cdot \frac{QP}{OP}$$

$= \cos NOQ \,.\, \cos QOP - \sin QPR \,.\, \sin QOP$

$= \cos A \,.\, \cos B - \sin A \,.\, \sin B.$

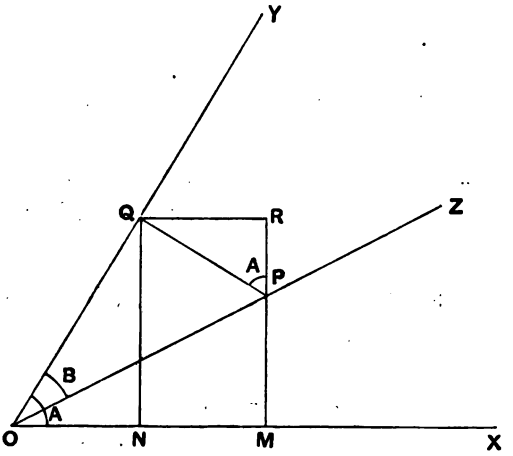
Since the triangles NOQ, RPQ are similar,

$$\therefore \frac{RP}{ON} = \frac{QP}{OQ} = \tan B,$$

(3) $\tan(A+B) = \tan XOZ = \dfrac{MP}{OM} = \dfrac{NQ+RP}{ON-RQ}$

$$= \frac{\dfrac{NQ}{ON} + \dfrac{RP}{ON}}{\dfrac{ON}{ON} - \dfrac{RQ}{RP} \cdot \dfrac{RP}{ON}} = \frac{\tan A + \tan B}{1 - \tan A \,.\, \tan B}.$$

330. *To find the sine, cosine, and tangent* of the difference of two angles *in terms of the ratios of the angles.*



Let $XOY = A$, and $ZOY = B$. Then $XOZ = A - B$.

In OZ, *the final line of the compound angle* $A - B$, take any point P.

From P draw PM, PQ perpendiculars upon OX, OY—*the initial and final lines of* A—respectively.

From Q draw QN, QR perpendiculars upon OX, MP respectively.

Then $\angle QPR = 90° - \angle RQP = \angle RQY = \angle XOQ = A$.

[Or: RP, QP being perpendiculars upon OX, OY contain the same angle as OX, OY.]

(1) $\sin(A-B) = \sin XOZ = \dfrac{MP}{OP} = \dfrac{MR-PR}{OP} = \dfrac{NQ}{OP} - \dfrac{PR}{OP}$

= [introducing required hyps. under the numerators]

$$\frac{NQ}{OQ} \cdot \frac{OQ}{OP} - \frac{PR}{PQ} \cdot \frac{PQ}{OP}$$

$= \sin NOQ \,.\cos POQ - \cos QPR \,.\sin POQ$

$= \sin A \,.\cos B - \cos A \,.\sin B.$

(2) $\cos(A-B) = \cos XOZ = \dfrac{OM}{OP} = \dfrac{ON+NM}{OP} = \dfrac{ON}{OP} + \dfrac{QR}{OP}$

= [introducing required hyps. under the numerators]

$$\frac{ON}{OQ}\cdot\frac{OQ}{OP} + \frac{QR}{PQ}\cdot\frac{PQ}{OP}$$

$= \cos NOQ \,.\, \cos POQ + \sin QPR \,.\, \sin POQ$

$= \cos A \,.\, \cos B + \sin A \,.\, \sin B.$

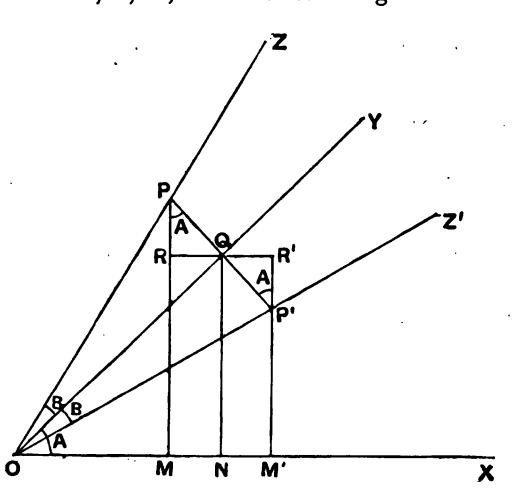
Since the triangles NOQ, RPQ are similar,

$$\therefore \frac{PR}{ON} = \frac{PQ}{OQ} = \tan B,$$

(3) $\tan(A-B) = \tan XOZ = \dfrac{MP}{OM} = \dfrac{NQ-PR}{ON+QR}$

$$= \frac{\dfrac{NQ}{ON} - \dfrac{PR}{ON}}{\dfrac{ON}{ON} + \dfrac{QR}{PR}\cdot\dfrac{PR}{ON}} = \frac{\tan A - \tan B}{1+\tan A \tan B}.$$

331. We may combine the above two figures into one, if we dash the letters Z, P, M, R in the second figure. Thus



Let $XOY = A$, and $YOZ = Z'OY = B$.

Then $XOZ = A + B$, and $XOZ' = A - B$.

In OY take any point Q; through Q draw PQP' perpendicular to OY, cutting OZ, OZ' in P, P'.

Draw QN, PM, $P'M'$ perpendiculars on OX;

and RQR' perpendicular to QN, cutting PM, $P'M'$ in R, R'.

Then $OP = OP'$, $RQ = QR'$, $RP = P'R'$.

Also $\angle QP'R' = \angle QPR = 90^\circ - \angle RQP = \angle OQR = A$,

$$\sin(A+B) + \sin(A-B) = \sin XOZ + \sin XOZ' = \frac{MP}{OP} + \frac{M'P'}{OP'}$$

$$= \frac{(NQ+RP)+(NQ-P'R')}{OP} = \frac{2.NQ}{OP} = \frac{2.NQ}{OQ} \cdot \frac{OQ}{OP} = 2\sin A \cos B.$$

$$\sin(A+B) - \sin(A-B) = \sin XOZ - \sin XOZ' = \frac{MP}{OP} - \frac{M'P'}{OP'}$$

$$= \frac{(NQ+RP)-(NQ-P'R')}{OP} = \frac{2.RP}{OP} = \frac{2.RP}{QP} \cdot \frac{QP}{OP} = 2\cos A \sin B.$$

$$\cos(A-B) + \cos(A+B) = \cos XOZ' + \cos XOZ = \frac{OM'}{OP'} + \frac{OM}{OP}$$

$$= \frac{(ON+QR')+(ON-RQ)}{OP} = \frac{2.ON}{OP} = \frac{2.ON}{OQ} \cdot \frac{OQ}{OP} = 2\cos A \cos B.$$

$$\cos(A-B) - \cos(A+B) = \cos XOZ' - \cos XOZ = \frac{OM'}{OP'} - \frac{OM}{OP}$$

$$= \frac{(ON+QR')-(ON-RQ)}{OP} = \frac{2.RQ}{OP} = \frac{2.RQ}{QP} \cdot \frac{QP}{OP} = 2\sin A \sin B.$$

332. Taking the angle XOZ to be P, and XOZ' to be Q in the above figure, from Art. 116, we see that

$$XOY = \tfrac{1}{2}(XOZ + XOZ') = \tfrac{1}{2}(P+Q)$$

and $$YOZ = \tfrac{1}{2}(XOZ - XOZ') = \tfrac{1}{2}(P-Q).$$

Hence the above formulæ take the form

$$\sin P + \sin Q = 2\sin \tfrac{1}{2}(P+Q)\cos \tfrac{1}{2}(P-Q),$$
$$\sin P - \sin Q = 2\cos \tfrac{1}{2}(P+Q)\sin \tfrac{1}{2}(P-Q),$$
$$\cos Q + \cos P = 2\cos \tfrac{1}{2}(P+Q)\cos \tfrac{1}{2}(P-Q),$$
$$\cos Q - \cos P = 2\sin \tfrac{1}{2}(P+Q)\sin \tfrac{1}{2}(P-Q).$$

Extension of formulæ to all *angles.*

333. (1) *To extend the* $A+B$ *formulæ to* all positive acute *values of* A *and* B.

Let A and B be positive acute angles; but $A+B$ *obtuse.*

Let $A = 90° - C$; and $B = 90° - D$.

Then $A+B = 180° - (C+D)$, $\therefore$ $C+D$ is *acute.*

Now $\sin(A+B)$ i.e. $\sin\{180° - (C+D)\}$

$= \sin(C+D)$, by Art. 308,

$= \sin C \cos D + \cos C \sin D$, by Art. 329,

$= \cos(90° - C)\sin(90° - D) + \sin(90° - C)\cos(90° - D)$, by Art. 76,

i.e. $\cos A \sin B + \sin A \cos B$. Q. E. D.

And $\cos(A+B)$ i.e. $\cos\{180° - (C+D)\}$

$= -\cos(C+D)$, by Art. 308,

$= -\cos C \cos D + \sin C \sin D$, by Art. 329,

$= -\sin(90° - C)\sin(90° - D) + \cos(90° - C)\cos(90° - D)$, by Art. 76.

i.e. $-\sin A \sin B + \cos A \cos B$. Q. E. D.

334. (2) *To extend the* $A+B$ *formulæ to all* positive *values of* A *and* B.

Assume that for some particular two values of A and B,

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \quad \ldots\ldots(1)$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B \quad \ldots\ldots(2).$$

Let $B' = B + 90°$.

Then

$\sin(A+B')$ i.e. $\sin\{A + (B+90°)\} = \cos(A+B)$, by Art. 326,

$= \cos A \cos B - \sin A \sin B$, by assumption,

$= \cos A \sin(90° + B) + \sin A \cos(90° + B)$, by Art. 326,

i.e. $\cos A \sin B' + \sin A \cos B'$.

And

$\cos(A + B')$ i.e. $\cos\{A + (B + 90°)\} = -\sin(A + B)$, by Art. 326,

$= -\sin A \cos B - \cos A \sin B$, by assumption,

$= -\sin A \sin(90° + B) + \cos A \cos(90° + B)$, by Art. 326,

i.e. $-\sin A \sin B' + \cos A \cos B'$.

Thus, *assuming* the formulæ (1) and (2) for any *given* values of A and B, we have proved them true for values A and $B + 90°$.

But they have been proved in last article to be true for all *positive acute angles.*

Hence they are true for all *positive* angles of any magnitude.

335. (3) *To extend the $A + B$ formulæ to* all *values of A and B positive or* negative.

Assuming the formulæ true for some particular two values of A and B, we may prove precisely as in the last article that they are true for A and $B - 90°$, by means of the formulæ of Art. 326,

$$\sin(B - 90°) = -\cos B \text{ and } \cos(B - 90°) = +\sin B.$$

Hence, by induction, the formulæ hold for all *negative* (as well as positive) values of A and B.

336. (4) *To prove the $A - B$ formulæ from the $A + B$ formulæ.*

$\sin(A - B)$ i.e. $\sin\{A + (-B)\}$

$= \sin A \cos(-B) + \cos A \sin(-B)$

$= \sin A \cos B - \cos A \sin B$ by Art. 308,

$\cos(A - B)$ i.e. $\cos\{A + (-B)\}$

$= \cos A \cos(-B) - \sin A \sin(-B)$

$= \cos A \cos B + \sin A \sin B$ by Art. 308.

Hence the $A - B$ formulæ follow from the $A + B$ formulæ, for any values of A and B.

337. Another method of extending the proof of the formulæ of Arts. 329, 330 to angles of any sign or magnitude is to substitute *directed lengths* for *arithmetic lengths* in those proofs.

The order of lettering has been so chosen that no change need be introduced in the general proofs, except that the lengths are to be regarded as directed lengths.

Now as far as the original angles A, B, $A+B$, $A-B$ are concerned, all the equations are seen by inspection to be true from the general definitions of ratios, because of the fundamental formula for directed lengths $(LM)+(MN)=(LN)$.

But it is *not* evident by inspection that in all cases the following equations hold in *sign* as well as in magnitude, viz.

$$\frac{(RP)}{(QP)}=\cos A \text{ and } \frac{(RQ)}{(QP)}=\sin A.$$

The student may, by drawing appropriate figures, work out for himself some of the 64 cases which arise, according to the different values of A, B, $A+B$, $A-B$.

But a more general proof will be given below.

338. In comparing the $A-B$ with the $A+B$ proofs the student should observe that since $(RP)=-(PR)$ and so on, the proofs are exactly equivalent, except that, since in the $(A+B)$ case, $B=(YOZ)$ or (QOP); but in the $A-B$ case, $B=(ZOY)$ or (POQ), we have

in the former, $$\sin B=\frac{(QP)}{(OP)},$$

in the latter, $$\sin B=\frac{(PQ)}{(OP)}.$$

In each case, of course $\sin(QOP)=\dfrac{(QP)}{(OP)}$,

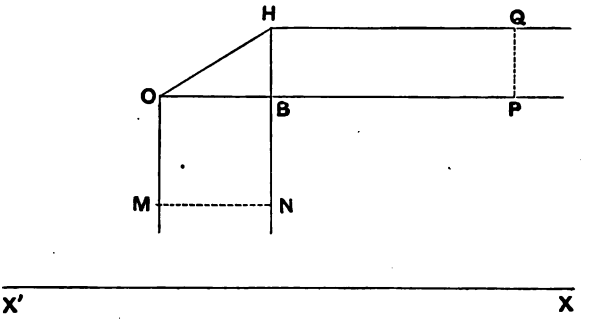
and $$\sin(POQ)=\frac{(PQ)}{(OP)}.$$

The two cases are reduced to one case, if we always take (YOZ) to be B, and apply it to *negative* (as well as positive) values of B.

§ 2. Projections and Erections*.

339. Def. 1. The **projection** of a directed length on an indefinite line is the directed distance between the two *perpendiculars* to the indefinite line drawn through the extremities of the directed length.

Def. 2. The **erection** of a directed length on an indefinite line is the directed distance between the two *parallels* to the indefinite line drawn through the extremities of the directed length.



Thus, if OH is any directed length, and $X'X$ an indefinite line, and if OM, HBN be perpendiculars upon $X'X$, and if HQ, OBP be parallels to $X'X$ (see figure) then

(MN) or (OB) is the *projection* of (OH) upon $X'X$; and

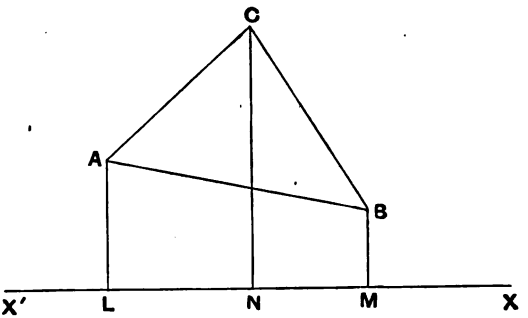
(PQ) or (BH) is the *erection* of (OH) upon $X'X$.

The *projection* is thus *parallel* to the indefinite line; and the *erection* is *perpendicular* to it.

It is clear that the *erection* upon one line is the *projection* upon any *perpendicular* to it; and *vice versa*.

* The definitions of these terms are given here on the supposition that all the lines, to which reference is made, are in *one plane*.

340. Prop. I. If A, B, C be any three points and $X'X$ any line, the projection (or erection) of (AC) upon $X'X$ is equal to the sum of the projections (or erections) of (AB) and (BC) upon $X'X$.



For, let AL, BM, CN be perpendiculars upon $X'X$.

Then, however L, M, N may fall, we have

$$(LN) = (LM) + (MN),$$

i.e. projection of (AC) = projection of (AB) + projection of (BC).

Since the erection is merely the projection on a perpendicular, the same proposition holds of erections.

341. Prop. II. The sum of the projections or erections of the sides of any closed figure, directed from point to point round the figure, is zero.

For, let $ABCDEF$ be such a closed figure.

Then, by Prop. I.,

proj. of (AB) + proj. of (BC) = proj. of (AC)

and proj. of (AC) + proj. of (CD) = proj. of (AD)

and proj. of (AD) + proj. of (DE) = proj. of (AE)

and so on.

Finally

proj. of (AF) + proj. of (FA) = zero.

$\therefore$ proj. of (AB) + proj. of (BC) + ... + proj. of (FA) = zero.

The same clearly holds of erections.

342. *The* projection *and* erection *of any directed length upon any indefinite line are obtained by multiplying the directed length by the* cosine *and* sine *respectively of the angle which the positive direction along the* directed length *makes with the positive direction along the* indefinite line.

For consider a line to revolve round O from its (positive) initial direction parallel to $X'X$ or XX' to its (positive) final direction along OH or HO (produced), and so describe an angle θ. [See fig. Art. 339.]

Then (OB), (OH)—directed from O—are respectively *base* and *hypothenuse.*

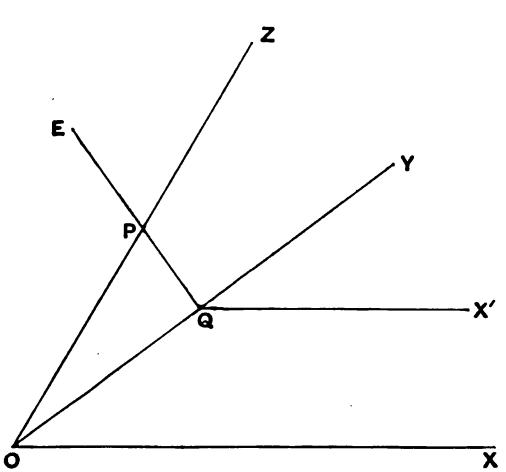
And (BH)—directed from the initial line—is the *perpendicular.*

Thus

$$\frac{(OB)}{(OH)} = \cos\theta \text{ and } \frac{(BH)}{(OH)} = \sin\theta.$$

$$\therefore (OB) = (OH)\cos\theta \text{ and } (BH) = (OH)\sin\theta.$$

343. *To find the* cosine *and* sine *of the* sum *of two angles by projections and erections.*



Let (XOY) be any angle A; (YOZ) any angle B; then $(XOZ) = A + B$.

In (OY) take any point Q: through Q draw QX' *in the direction* OX, and QE *making a positive right-angle with* OY.

Then $\angle X'QE$ is *always* $90° + X'QY$ i.e. $90° + A$.

Let QE, OZ—either being produced, if necessary—cut in P. Then, by Art. 340,

proj. of (OP) = proj. of (OQ) + proj. of (QP),

and erec. of (OP) = erec. of (OQ) + erec. of (QP).

Taking these projections and erections upon OX,

$$(OP)\cos(XOZ) = (OQ)\cos(XOY) + (QP)\cos(X'QE),$$

and $$(OP)\sin(XOZ) = (OQ)\sin(XOY) + (QP)\sin(X'QE).$$

But (OQ), (QP) are, respectively, the projection and erection of (OP) upon (OY).

$$\therefore (OQ) = (OP)\cos(YOZ) \text{ and } (QP) = (OP)\sin(YOZ).$$

Substituting and dividing by (OP) we have

$$\cos(A + B) = \cos B \,.\, \cos A + \sin B \,.\, \cos(90° + A),$$
$$\sin(A + B) = \cos B \,.\, \sin A + \sin B \,.\, \sin(90° + A).$$

But, by Art. 326,

$$\cos(90° + A) = -\sin A \text{ and } \sin(90° + A) = +\cos A,$$

$$\therefore \quad \cos(A + B) = \cos A \cos B - \sin A \sin B,$$

and $$\sin(A + B) = \sin A \cos B + \cos A \sin B.$$

The above proof is perfectly general.

Hence, writing $-B$ for B,

since $\cos(-B) = +\cos B$ and $\sin(-B) = -\sin B$,

$$\therefore \quad \cos(A - B) = \cos A \cos B + \sin A \sin B,$$

and $$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$

344. The student should observe that the important triangular formulæ

$$c \sin B = b \sin C,$$
$$c \cos B + b \cos C = a,$$

are obtained simply by erecting and projecting upon BC.

§ 3. Extension to Three or more Angles.

345. *To find the sine, cosine, and tangent of the sum of* three *angles.*

(i) $\sin(A+B+C)$ i.e. $\sin\{(A+B)+C\}$

$$= \sin(A+B)\cos C + \cos(A+B)\sin C$$

$$= \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C.$$

(ii) $\cos(A+B+C)$ i.e. $\cos\{(A+B)+C\}$

$$= \cos(A+B)\cos C - \sin(A+B)\sin C$$

$$= \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C.$$

(iii) $\tan(A+B+C)$ i.e. $\tan\{(A+B)+C\}$

$$= \frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}$$

$$= \frac{\dfrac{\tan A+\tan B}{1-\tan A\tan B}+\tan C}{1-\dfrac{(\tan A+\tan B)\tan C}{1-\tan A\tan B}}$$

$$= \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan A\tan C-\tan B\tan C}.$$

346. These formulæ should be carefully noted and their symmetry examined. In the same way we may prove formulæ involving three angles one or more of which is subtracted.

347. The following results should be noticed:—

$$(\cos A+\sin A)(\cos B+\sin B)$$

$$= \cos A\cos B + \sin A\cos B + \cos A\sin B + \sin A\sin B.$$

Here first term − last term $= \cos(A+B)$,

and the middle terms $= \sin(A+B)$.

Again $(1+\tan A)(1+\tan B)$

$$= 1+\tan A+\tan B+\tan A\tan B.$$

Here first term − last term = den. of $\tan(A+B)$,

and the middle terms = num. of $\tan(A+B)$.

348. Again

$$(\cos A + \sin A)(\cos B + \sin B)(\cos C + \sin C)$$
$$= (\cos A \cos B \cos C)$$
$$+ (\sin A \cos B \cos C + \sin B \cos C \cos A + \sin C \cos A \cos B)$$
$$+ (\cos A \sin B \sin C + \cos B \sin C \sin A + \cos C \sin A \sin B)$$
$$+ (\sin A \sin B \sin C).$$

Here, arranging the terms in groups according to the number of *sines* in each term, we have

$$\text{first group} - \text{third group} = \cos(A + B + C),$$
$$\text{second group} - \text{fourth group} = \sin(A + B + C).$$

349. The result suggested by the last two articles may be shown to be true for any number of angles.

350. Arrange the product

$$(\cos A + \sin A)(\cos B + \sin B)(\cos C + \sin C)\ldots$$

to any number of factors, in groups of terms, according to the ascending number of sines in each term.

Then the expansions of

$$\cos(A + B + C + \ldots) \text{ and of } \sin(A + B + C + \ldots)$$

will consist respectively of the above groups, *taken alternately* but *with alternate sign.*

In other words, if the product of n factors

$$(\cos A + \sin A)(\cos B + \sin B)\ldots = S_0 + S_1 + \ldots + S_r + \ldots + S_n \quad (1),$$

wherein each term of the group S_r contains r sines, then

$$\cos(A + B + \ldots) = S_0 - S_2 + S_4 - \ldots\ldots\ldots\ldots\ldots (2),$$
$$\sin(A + B + \ldots) = S_1 - S_3 + S_5 - \ldots\ldots\ldots\ldots\ldots (3).$$

For assume this to be true for n angles.

Introduce an $(n+1)^{\text{th}}$ angle, X say. Then, by multiplication of (1),

$$(\cos A + \sin A)(\cos B + \sin B)\ldots(\cos X + \sin X)$$
$$= (S_0 + S_1 + \ldots + S_n)\cos X + (S_0 + S_1 + \ldots + S_n)\sin X$$
$$= S_0 \cos X + (S_1 \cos X + S_0 \sin X) + (S_2 \cos X + S_1 \sin X)$$
$$+ (S_3 \cos X + S_2 \sin X) + \ldots\ldots\ldots (1'),$$

where the terms are grouped according to the number of their sines.

But

$$\cos(A+B+\ldots+X) = \cos(A+B+\ldots)\cos X - \sin(A+B+\ldots)\sin X$$
$$= (S_0 - S_2 + S_4 - \ldots)\cos X - (S_1 - S_3 + S_5 - \ldots)\sin X$$
$$= S_0\cos X - (S_2\cos X + S_1\sin X) + (S_4\cos X + S_3\sin X) - \ldots(2'),$$

by (2) and (3).

And

$$\sin(A+B+\ldots+X) = \sin(A+B+\ldots)\cos X + \cos(A+B+\ldots)\sin X$$
$$= (S_1 - S_3 + S_5 - \ldots)\cos X + (S_0 - S_2 + S_4 - \ldots)\sin X$$
$$= (S_1\cos X + S_0\sin X) - (S_3\cos X + S_2\sin X) + \ldots\ldots(3'),$$

by (2) and (3).

Comparing (1'), (2'), (3') we see that the proposition holds for $(n+1)$ angles, if it is assumed for n. Now it has been seen to be true for 2 or for 3 angles, $\therefore$ it is true for *any* number of angles.

351. In the same way the student may show that if the product $(1+\tan A)(1+\tan B)(1+\tan C)\ldots$ to any number of factors is arranged in ascending groups of terms according to the number of tangents in each term, then the den. and num. of $\tan(A+B+C+\ldots)$ will consist respectively of the above groups, *taken alternately* but *with alternate sign*. In other words

If $(1+\tan A)(1+\tan B)\ldots = 1 + T_1 + T_2 + \ldots + T_r + \ldots + T_n$,

then $$\tan(A+B+C+\ldots) = \frac{T_1 - T_3 + T_5 - \ldots}{1 - T_2 + T_4 - \ldots}.$$

Similarly,

if $(\cot A + 1)(\cot B + 1)\ldots = K_0 + K_1 + K_2 + \ldots + K_r + \ldots + 1$,

where K_r denotes the group of terms containing $n-r$ cotangents,

then $$\cot(A+B+C+\ldots) = \frac{K_0 - K_2 + K_4 - \ldots}{K_1 - K_3 + K_5 - \ldots}.$$

EXAMPLES XIII.

1. $\sqrt{3} + \tan 40° + \tan 80° = \sqrt{3} \cdot \tan 40° \cdot \tan 80°$.

2. $\tan 20° + \tan 40° + \sqrt{3} \tan 20° \tan 40° = \sqrt{3}$.

3. $\cos A \cos (B + C) - \cos B \cos (A + C) = \sin (A - B) \sin C$.

4. $\sin A \cos (B + C) - \sin B \cos (A + C) = \sin (A - B) \cos C$.

5. If $A + B + C$ is an odd multiple of π
$$\sin^2 B + \sin^2 C = \sin^2 A + 2 \cos A \sin B \sin C.$$

6. $$\frac{\tan (n+1)\phi - \tan n\phi}{1 + \tan (n+1)\phi \cdot \tan n\phi} = \tan \phi.$$

7. $\cos (\alpha + \beta) + \sin (\alpha - \beta) = 2 \sin (\frac{1}{4}\pi + \alpha) \cos (\frac{1}{4}\pi + \beta)$.

8. $$\sin \alpha + \sin \beta + \sin \gamma - \sin (\alpha + \beta + \gamma) = 4 \sin \tfrac{1}{2} (\beta + \gamma) \sin \tfrac{1}{2} (\gamma + \alpha) \sin \tfrac{1}{2} (\alpha + \beta).$$

9. $$\cos \alpha + \cos \beta + \cos \gamma + \cos (\alpha + \beta + \gamma) = 4 \cos \tfrac{1}{2} (\beta + \gamma) \cos \tfrac{1}{2} (\gamma + \alpha) \cos \tfrac{1}{2} (\alpha + \beta).$$

10. $$\sin \alpha + \sin \beta - \sin \gamma - \sin (\alpha + \beta - \gamma) = 4 \sin \tfrac{1}{2} (\alpha - \gamma) \sin \tfrac{1}{2} (\beta - \gamma) \sin \tfrac{1}{2} (\alpha + \beta).$$

11. $$\cos \alpha + \cos \beta + \cos \gamma + \cos (\alpha + \beta - \gamma) = 4 \cos \tfrac{1}{2} (\beta - \gamma) \cos \tfrac{1}{2} (\alpha - \gamma) \cos \tfrac{1}{2} (\alpha + \beta).$$

12. $$\sin (\beta - \gamma) + \sin (\gamma - \alpha) + \sin (\alpha - \beta) = -4 \sin \tfrac{1}{2} (\beta - \gamma) \sin \tfrac{1}{2} (\gamma - \alpha) \sin \tfrac{1}{2} (\alpha - \beta).$$

13. $$\sin (\beta + \gamma - \alpha) + \sin (\gamma + \alpha - \beta) + \sin (\alpha + \beta - \gamma) - \sin (\alpha + \beta + \gamma) = 4 \sin \alpha \sin \beta \sin \gamma.$$

14. $$\cos (\beta + \gamma - \alpha) + \cos (\gamma + \alpha - \beta) + \cos (\alpha + \beta - \gamma) + \cos (\alpha + \beta + \gamma) = 4 \cos \alpha \cos \beta \cos \gamma.$$

15. $$\cos^2 x + \cos^2 y + \cos^2 z + \cos^2 (x + y + z) = 2 \{1 + \cos (y + z) \cos (z + x) \cos (x + y)\}.$$

16. $$\sin^2 x + \sin^2 y + \sin^2 z + \sin^2 (x + y + z) = 2 \{1 - \cos (y + z) \cos (z + x) \cos (x + y)\}.$$

17. $\cos\alpha\sin(\beta-\gamma)+\cos\beta\sin(\gamma-\alpha)+\cos\gamma\sin(\alpha-\beta)=0.$

18. $\sin\alpha\sin(\beta-\gamma)+\sin\beta\sin(\gamma-\alpha)+\sin\gamma\sin(\alpha-\beta)=0.$

19. $\sin(\alpha+\beta+\gamma+\delta)$
$=\cos\alpha\cos\beta\cos\gamma\cos\delta(\tan\alpha+\tan\beta+\tan\gamma+\tan\delta)$
$-\sin\alpha\sin\beta\sin\gamma\sin\delta(\cot\alpha+\cot\beta+\cot\gamma+\cot\delta).$

20. $\cos(S-A)\cos(S-B)\cos(S-C)\cos(S-D)$
$+\sin(S-A)\sin(S-B)\sin(S-C)\sin(S-D)$
$=\cos A\cos B\cos C\cos D+\sin A\sin B\sin C\sin D$
if $2S=A+B+C+D.$

21. If a right-angle is divided into three parts whose tangents are $l\rho$, $m\rho$, $n\rho$;
then $\rho^{-2}=mn+nl+lm.$

22. $\tan(A+B)+\tan(A-B)=\dfrac{2\tan A\sec^2 B}{1-\tan^2 A\tan^2 B}.$

23. $\tan(A+B)-\tan(A-B)=\dfrac{2\tan B\sec^2 A}{1-\tan^2 A\tan^2 B}.$

24. $\cot(A-B)+\cot(A+B)=\dfrac{2\cot A\operatorname{cosec}^2 B}{\cot^2 B-\cot^2 A}.$

25. $\cot(A-B)-\cot(A+B)=\dfrac{2\cot B\operatorname{cosec}^2 A}{\cot^2 B-\cot^2 A}.$

26. $\sec(A+B)+\sec(A-B)=\dfrac{2\sec A\sec B}{\sec^2 A+\sec^2 B-\sec^2 A\sec^2 B}.$

27. $\sec(A+B)-\sec(A-B)=\dfrac{2\operatorname{cox} A\operatorname{cox} B}{\operatorname{cox}^2 A\operatorname{cox}^2 B-\operatorname{cox}^2 A-\operatorname{cox}^2 B}.$

If A, B, C are the angles of a triangle, prove 28—33:

28. $\sin A\sin B\sin C$
$=\sin A\cos B\cos C+\sin B\cos C\cos A+\sin C\cos A\cos B.$

29. $1+\cos A\cos B\cos C$
$=\cos A\sin B\sin C+\cos B\sin C\sin A+\cos C\sin A\sin B.$

30. $\tan A\tan B\tan C=\tan A+\tan B+\tan C.$

31. $\cot B\cot C+\cot C\cot A+\cot A\cot B=1.$

32. $\cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C = \sin \frac{1}{2}A \sin \frac{1}{2}B \cos \frac{1}{2}C$
$+ \sin \frac{1}{2}B \sin \frac{1}{2}C \cos \frac{1}{2}A + \sin \frac{1}{2}C \sin \frac{1}{2}A \cos \frac{1}{2}B.$

33. $\tan \frac{1}{2}B \tan \frac{1}{2}C + \tan \frac{1}{2}C \tan \frac{1}{2}A + \tan \frac{1}{2}A \tan \frac{1}{2}B = 1.$

34. If A, B, C, D are the angles of a quadrilateral,

(i) $\dfrac{\tan A + \tan B + \tan C + \tan D}{\cot A + \cot B + \cot C + \cot D} = \tan A \tan B \tan C \tan D,$

(ii) $\sin A \sin B \cos C \cos D + \ldots$
$= \cos A \cos B \cos C \cos D + \sin A \sin B \sin C \sin D - 1.$

35. If θ, ϕ, ψ are the angles which any straight line, in the plane of the triangle ABC, makes with BC, CA, AB respectively, then

(i) $a^2 \sin(\phi + \psi - \theta) + b^2 \sin(\psi + \theta - \phi) + c^2 \sin(\theta + \phi - \psi)$
$+ 2bc \sin\theta + 2ca \sin\phi + 2ab \sin\psi = 0.$

(ii) $a^2 \cos(\phi + \psi - \theta) + b^2 \cos(\psi + \theta - \phi) + c^2 \cos(\theta + \phi - \psi)$
$+ 2bc \cos\theta + 2ca \cos\phi + 2ab \cos\psi = 0.$

36. If $\sin\theta \sin\alpha \sin(\phi + \alpha - \beta) = \sin\phi \sin(\alpha + \beta) \sin(\alpha + \theta),$
then $\sin\phi \sin\alpha \sin(\theta + \alpha + \beta) = \sin\theta \sin(\alpha - \beta) \sin(\alpha + \phi).$

37. If $\cos\theta \cos\alpha \cos(\phi + \alpha - \beta) = \cos\phi \sin(\alpha + \beta) \sin(\alpha + \theta),$
then $\cos\phi \cos\alpha \cos(\theta + \alpha + \beta) = \cos\theta \sin(\alpha - \beta) \sin(\alpha + \phi).$

38. In a triangle right-angled at C,

$$\tan^{-1}\frac{a}{b+c} + \tan^{-1}\frac{b}{a+c} = (n + \tfrac{1}{4})\pi.$$

39. If $\sin^{-1} m + \sin^{-1} n = \frac{1}{2}\pi,$
then $m\sqrt{(1 - n^2)} + n\sqrt{(1 - m^2)} = 1.$

40. $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\dfrac{x + y}{1 - xy}.$

41. $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\dfrac{x - y}{1 + xy}.$

42. If $\dfrac{\cos(\alpha + \beta + \theta)}{\sin(\alpha + \beta)\cos^2\gamma} = \dfrac{\cos(\gamma + \alpha + \theta)}{\sin(\gamma + \alpha)\cos^2\beta}$

and if β and γ are unequal, then each member

$$= \frac{\cos(\beta+\gamma+\theta)}{\sin(\beta+\gamma)\cos^2\alpha},$$

and $$\cot\theta = \frac{\sin(\beta+\gamma)\sin(\gamma+\alpha)\sin(\alpha+\beta)}{\cos(\beta+\gamma)\cos(\gamma+\alpha)\cos(\alpha+\beta)+\sin^2(\alpha+\beta+\gamma)}.$$

43. If $\cos(\alpha+\beta+\gamma)-\sin(\alpha+\beta+\gamma)$
$$= 2\cos\alpha\cos\beta\cos\gamma + 2\sin\alpha\sin\beta\sin\gamma,$$
then either α or β or γ is of the form $(n-\frac{1}{4})\pi$.

44. If $\tan(\alpha+\theta)+\tan(\alpha-\theta) = \dfrac{2\sin\theta\sec^3\theta}{1-\tan^2\alpha\tan^2\theta}$,

then $\theta = n\pi \pm \alpha$ or $\frac{1}{2}(2n+1)\pi - \alpha$.

CHAPTER XIV.

MULTIPLE ANGLES.

352. *To find the ratios of* $2A$.

We have

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \quad \ldots\ldots\ldots(1),$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \quad \ldots\ldots\ldots(2),$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad \ldots\ldots\ldots(3).$$

In each of these formulæ, put $B = A$; thus

$$\sin(A+A) = \sin A \cos A + \cos A \sin A,$$

i.e. $$\sin 2A = 2 \sin A \cos A \quad \ldots\ldots\ldots(4).$$

$$\cos(A+A) = \cos A \cos A - \sin A \sin A,$$

i.e. $$\cos 2A = \cos^2 A - \sin^2 A \quad \ldots\ldots\ldots(5).$$

$$\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A},$$

i.e. $$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \quad \ldots\ldots\ldots(6).$$

353. Now we may observe that

$$(\cos A + \sin A)^2 = \cos^2 A + 2 \sin A \cos A + \sin^2 A,$$

$$(1 + \tan A)^2 \quad = 1 + 2 \tan A + \tan^2 A.$$

Comparing these expansions with the above formulæ (4), (5), (6), we see that

first term – last term of $(\cos A + \sin A)^2 = \cos 2A$,
middle term of $(\cos A + \sin A)^2 = \sin 2A$,

and first term – last term of $(1 + \tan A)^2 =$ den. of $\tan 2A$,
middle term of $(1 + \tan A)^2 =$ num. of $\tan 2A$.

354. *To find the ratios of* $3A$.

In (1), (2), (3) of Art. 352, put $B = 2A$; thus

$$\sin(A + 2A) = \sin A \cos 2A + \cos A \sin 2A,$$

i.e. by (4) and (5)

$$\sin 3A = \sin A (\cos^2 A - \sin^2 A) + \cos A \,.\, 2 \sin A \cos A$$
$$= 3 \sin A \cos^2 A - \sin^3 A \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

$$\cos(A + 2A) = \cos A \cos 2A - \sin A \sin 2A,$$

i.e. by (4) and (5)

$$\cos 3A = \cos A (\cos^2 A - \sin^2 A) - \sin A \,.\, 2 \sin A \cos A$$
$$= \cos^3 A - 3 \cos A \sin^2 A \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

$$\tan(A + 2A) = \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A},$$

i.e. by (6)

$$\tan 3A = \frac{\tan A + \dfrac{2 \tan A}{1 - \tan^2 A}}{1 - \dfrac{\tan A \,.\, 2 \tan A}{1 - \tan^2 A}},$$

$$= \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A} \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

355. As before, we here observe from (7), (8), (9), that

1st term − 3rd term of $(\cos A + \sin A)^3 = \cos 3A$,

2nd term − 4th term of $(\cos A + \sin A)^3 = \sin 3A$.

And

1st term − 3rd term of $(1 + \tan A)^3$ = den. of $\tan 3A$,

2nd term − 4th term of $(1 + \tan A)^3$ = num. of $\tan 3A$.

356. In the same way the student should show

that $$\cos 4A = \cos^4 A - 6 \cos^2 A \sin^2 A + \sin^4 A,$$

and $$\sin 4A = 4 \cos^3 A \sin A - 4 \cos A \sin^3 A,$$

and $$\tan 4A = \frac{4 \tan A - 4 \tan^3 A}{1 - 6 \tan^2 A + \tan^4 A}.$$

He should then compare these expressions with the expansions of $(\cos A + \sin A)^4$ and $(1 + \tan A)^4$.

357. The above cases suggest to us that we may write down the general values of $\cos nA$, $\sin nA$, and $\tan nA$ by means of the known expansion of any integral power of a binomial.

The general case follows at once from the formulæ proved in the last chapter for the ratios of the sum of any number of angles: but it will be useful to give an independent proof.

358. DEF. An algebraical expression consisting of terms, in each of which the *sum* of the *indices* of certain symbols is the same, is said to be **homogeneous** in respect to those symbols; and this constant sum is called the **dimension** of the expression.

Thus, $a^3 + 5abc + xa^2c$ is a homogeneous expression of three dimensions in a, b, c.

359. *To express $\cos nA$ and $\sin nA$ in a homogeneous series of n dimensions in $\cos A$ and $\sin A$.*

We will prove that

The terms of $\cos nA$ and $\sin nA$ are taken alternately, but with alternate change of sign, from the expansion of

$$(\cos A + \sin A)^n.$$

In other words,

If $$(\cos A + \sin A)^n = S_0 + S_1 + S_2 + \ldots + S_r + \ldots\ldots\ldots\text{(I)},$$

where S_r is the term involving $\sin^r A$, then

$$\cos nA = S_0 - S_2 + S_4 - \ldots\ldots\ldots\ldots\text{(II)}$$

and $$\sin nA = S_1 - S_3 + S_5 - \ldots\ldots\ldots\ldots\text{(III)}.$$

Assume this to be true for some particular value n.

Then, multiplying I by $\cos A + \sin A$,

$$(\cos A + \sin A)^{n+1} = (S_0 + S_1 + S_2 + \ldots\ldots)(\cos A + \sin A)$$
$$= S_0 \cos A + (S_1 \cos A + S_0 \sin A) + (S_2 \cos A + S_1 \sin A) + \ldots\text{(I')}.$$

in which the terms are grouped in powers of $\sin A$.

But $$\cos(n+1)A = \cos nA \cos A - \sin nA \sin A$$
$$= (S_0 - S_2 + S_4 - \ldots)\cos A - (S_1 - S_3 + S_5 - \ldots)\sin A$$
$$= S_0 \cos A - (S_2 \cos A + S_1 \sin A) + (S_4 \cos A + S_3 \sin A) - \ldots\text{(II')}.$$

And $\sin(n+1)A = \sin nA \cos A + \cos nA \sin A$

$= (S_1 - S_3 + S_5 - \ldots)\cos A + (S_0 - S_2 + S_4 - \ldots)\sin A$

$= (S_1 \cos A + S_0 \sin A) - (S_3 \cos A + S_2 \sin A) + \ldots\ldots\ldots\ldots(\text{III}')$.

Comparing I′, II′, and III′ we see that if the proposition is true for some particular value n it is true for the value $n+1$.

Now it is true for 2 and for 3; hence it is true for 4, for 5 &c. and therefore for any positive integral value of n.

Employing the known form of the binomial expansion, we have, *for any positive integral value of n*,

$$\cos nA = \cos^n A - \frac{n(n-1)}{1.2}\cos^{n-2} A \sin^2 A + \frac{n(n+1)(n-2)(n-3)}{1.2.3.4}\cos^{n-4} A \sin^4 A - \ldots$$

$$\sin nA = n\cos^{n-1} A \sin A - \frac{n(n-1)(n-2)}{1.2.3}\cos^{n-3} A \sin^3 A + \ldots$$

In the above important formulæ, it should be noted that the indices of $\sin A$ are *even* in the expansion of $\cos nA$ and *odd* in the expansion of $\sin nA$.

360. In the same way the student should prove that

$$\tan nA = \frac{n\tan A - \frac{1}{6}n(n-1)(n-2)\tan^3 A + \ldots}{1 - \frac{1}{2}n(n-1)\tan^2 A + \ldots}.$$

Or this formulæ follows at once from the expressions for $\sin nA$ and $\cos nA$, if we divide each by $\cos^n A$.

361. In Art. 359 we expressed the cosine and sine of any *multiple* of A in terms of the different *powers* of the cosine and sine of A. In the next two articles, we will do the converse problem: i.e. we will express any *power* of the cosine or sine of A in terms of the cosines or sines of the different *multiples* of A.

362. *To express* $\cos^n A$ *in a series of terms involving* cosines of multiples of A.

Let the symbol Σ denote the 'sum of all the terms obtained by taking every *alternative* that follows it.' Then, we have,

$$2\cos A = \cos A + \cos(-A) = \Sigma\cos(\pm A).$$

$$2\cos A \,.\, 2\cos B = 2\cos A\cos B + 2\cos(-A)\cos B$$
$$= \cos(A+B) + \cos(A-B) + \cos(-A+B) + \cos(-A-B)$$
$$= \Sigma\cos(\pm A \pm B).$$

Similarly

$$2\cos A \,.\, 2\cos B \,.\, 2\cos C = \Sigma\cos(\pm A \pm B) \,.\, 2\cos C$$
$$= \Sigma\cos(\pm A \pm B \pm C).$$

And universally

$$2\cos A \,.\, 2\cos B \,.\, 2\cos C \ldots 2\cos X = \Sigma\cos(\pm A \pm B \pm \ldots \pm X).$$

In this formula, put $A = B = C = \ldots = X$; and let there be n of these angles.

Then, on the right side, any term which arises from taking $n-r$ positive signs and r negative signs will be $\cos(n-2r)A$.

But the number of such terms is equal to the number of combinations of n things, r together. Hence

$$(2\cos A)^n = \cos nA + n\cos(n-2)A + \frac{n(n-1)}{1\,.\,2}\cos(n-4)A + \ldots\ldots$$
$$+ n\cos(-n+2)A + \cos(-nA)\ldots\ldots\ldots(1).$$

In this expansion it should be observed that the *coefficients* are the same as those of the binomial series $(1+x)^n$, and that there are thus $n+1$ terms.

The multiples of A descend by 2 from n to $-n$.

But since $\cos(-nA) = \cos nA$, and $\cos(-n+2)A = \cos(n-2)A$ and so on, we may express this series without using negative multiples of A. Thus $(2\cos A)^n$

$$= 2\cos nA + n\,.\,2\cos(n-2)A + \frac{n(n-1)}{1\,.\,2}\,.\,2\cos(n-4)A + \ldots(2),$$

in which every term involving a *positive* multiple of A is doubled but, if the term $\cos 0A$, i.e. 1, occurs it is *not doubled*.

363. In (1) writing for A, $\frac{1}{2}\pi - A$, since (Ex. XII., 39)

$$\cos n(\tfrac{1}{2}\pi - A) = (-1)^{\frac{n}{2}} \cos nA, \text{ if } n \text{ is even;}$$
$$= (-1)^{\frac{n-1}{2}} \sin nA, \text{ if } n \text{ is odd.}$$

$\therefore$ if n is even, $(-1)^{\frac{n}{2}}(2\sin A)^n$

$$= \cos nA - n\cos(n-2)A + \frac{n(n-1)}{1.2}\cos(n-4)A - \ldots\ldots$$

If n is odd, $(-1)^{\frac{n-1}{2}}(2\sin A)^n$

$$= \sin nA - n\sin(n+2)A + \frac{n(n-1)}{1.2}\sin(n-4)A - \ldots\ldots$$

Here the coefficients are those of the expansion $(1-x)^n$.

As before, negative multiples of A may be eliminated if we note that $\cos nA = +\cos(-nA)$ &c; and $\sin nA = -\sin(-nA)$.

364. In Art. 359, the cosine and sine of any multiple of an angle have been expressed in *homogeneous* series involving *both* the sine and cosine of the angle.

We will now, *as far as possible*, express them in terms of the sine or of the cosine *alone*.

This can be done by writing

$$1 - \cos^2 A \text{ for } \sin^2 A, \text{ and } 1 - \sin^2 A \text{ for } \cos^2 A.$$

365. Thus

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1$$
or $$= (1 - \sin^2 A) - \sin^2 A = 1 - 2\sin^2 A.$$

366. Again

$$\cos 3A = \cos^3 A - 3\cos A \sin^2 A = \cos^3 A - 3\cos A(1 - \cos^2 A)$$
$$= 4\cos^3 A - 3\cos A.$$
$$\sin 3A = 3\sin A \cos^2 A - \sin^3 A = 3\sin A(1 - \sin^2 A) - \sin^3 A$$
$$= 3\sin A - 4\sin^3 A.$$

These two formulæ are important.

367. Now we see generally that, since $\cos nA$ contains terms in which the index of $\sin A$ is always *even*, $\cos nA$ can always be expressed in powers of $\cos A$ alone; and, since $\sin nA$ contains terms in which the index of $\sin A$ is always *odd*, that $\sin nA/\sin A$ can also always be expressed in powers of $\cos A$ alone.

Similarly,

if n is even, both $\cos nA$ and $\sin nA/\cos A$;
if n is odd, both $\sin nA$ and $\cos nA/\cos A$

can be expressed in powers of $\sin A$ alone.

368. In order to find the required expressions we may make use of the important formulæ

$$\sin(n+1)A + \sin(n-1)A = 2\sin nA \,.\, \cos A \quad \ldots\ldots(1)$$
$$\cos(n-1)A + \cos(n+1)A = 2\cos nA \,.\, \cos A \quad \ldots\ldots(2)$$
$$\sin(n+1)A - \sin(n-1)A = 2\cos nA \,.\, \sin A \quad \ldots\ldots(3)$$
$$\cos(n-1)A - \cos(n+1)A = 2\sin nA \,.\, \sin A \quad \ldots\ldots(4).$$

The first two formulæ yield expressions in terms of $\cos A$; the last two in terms of $\sin A$.

369. For this purpose we will adopt the method of *recurring series.*

370. *To express* $\sin n\theta \div \sin\theta$ *in terms of* $2\cos\theta$.

Let $S =$ the sum of the infinite series whose general term is $x^r \sin r\theta$. Thus

$$S = x\sin\theta + x^2\sin 2\theta + \ldots + x^{n+1}\sin(n+1)\theta + \ldots\ldots$$
$$\therefore\ S \,.\, 2x\cos\theta = 2x^2\sin\theta\cos\theta + \ldots + 2x^{n+1}\sin n\theta\cos\theta + \ldots\ldots$$
$$\text{and}\qquad S \,.\, x^2 = x^2\sin\theta + \ldots \qquad + x^{n+1}\sin(n-1)\theta + \ldots\ldots$$

Taking the 1st − 2nd + 3rd of these equations,

$$S(1 - 2x\cos\theta + x^2) = x\sin\theta$$

for the higher powers of x vanish by (1) of Art. 368.

$$\therefore\qquad S = x\sin\theta\,(1 - 2x\cos\theta + x^2)^{-1}.$$

Now, by division (or by the binomial theorem),

$$(1 - 2x\cos\theta + x^2)^{-1} = \{1 - x(2\cos\theta - x)\}^{-1}$$
$$= 1 + x(2\cos\theta - x) + \ldots\ldots$$
$$+ x^{n-2}(2\cos\theta - x)^{n-2} + x^{n-1}(2\cos\theta - x)^{n-1} + \ldots\ldots$$

Hence, equating the coefficients of x^n in the two expressions for S, we have $\sin n\theta \div \sin\theta$

$$= (2\cos\theta)^{n-1} - (n-2)(2\cos\theta)^{n-3} + \frac{(n-3)(n-4)}{1.2}(2\cos\theta)^{n-5} - \ldots$$

as long as the index of $2\cos\theta$ is not negative.

371. *To express* $2\cos n\theta$ *in terms of* $2\cos\theta$.

Let

$$S = 1 + 2x\cos\theta + 2x^2\cos 2\theta + \ldots + 2x^{n+1}\cos(n+1)\theta + \ldots$$

$$\therefore S \,.\, 2x\cos\theta = 2x\cos\theta + 4x^2\cos^2\theta + \ldots + 4x^{n+1}\cos n\theta\cos\theta + \ldots$$

and $S \,.\, x^2 = \qquad x^2 + \ldots \qquad + 2x^{n+1}\cos(n-1)\theta + \ldots$

$$\therefore \qquad S(1 - 2x\cos\theta + x^2) = 1 - x^2$$

for the higher powers of x vanish by (2), of Art. 368,

$$\therefore \qquad S = (1-x^2)(1-2x\cos\theta+x^2)^{-1}.$$

$\therefore$ coefficient of x^n in S

$$= (\text{coeff. of } x^n - \text{coeff. of } x^{n-2}) \text{ in } (1 - 2x\cos\theta + x^2)^{-1}.$$

Now as in last article, $(1 - 2x\cos\theta + x^2)^{-1}$

$$= 1 + x(2\cos\theta - x) + \ldots + x^{n-2}(2\cos\theta - x)^{n-2} + x^{n-1}(2\cos\theta - x)^{n-1} + \ldots$$

Here the coefficient of x^n

$$= (2\cos\theta)^n - (n-1)(2\cos\theta)^{n-2} + \ldots + (-1)^r \frac{(n-r)(n-r-1)\ldots(n-2r+1)}{\underline{|r}}(2\cos\theta)^{n-2r} + \ldots$$

Writing $n-2$ instead of n, and $r-1$ instead of r, we have the coefficient of x^{n-2}

$$= (2\cos\theta)^{n-2} - (n-3)(2\cos\theta)^{n-4} + \ldots + (-1)^{r-1} \frac{(n-r-1)(n-r-2)\ldots(n-2r+1)}{\underline{|r-1}}(2\cos\theta)^{n-2r} + \ldots$$

Thus
$$2\cos n\theta$$
$$= (2\cos\theta)^n - n(2\cos\theta)^{n-2} + \ldots$$
$$+ (-1)^r n \cdot \frac{(n-r-1)(n-r-2)\ldots(n-2r+1)}{\lfloor r} (2\cos\theta)^{n-2r} + \ldots$$
as long as the index of $2\cos\theta$ is not negative.

372. In the last two articles, we have expressed the series in *descending* powers of $2\cos\theta$. We may now express them in *ascending* powers.

373. Thus, by Art. 370,

$\sin n\theta \div \sin\theta =$ coeff. of x^{n-1} in $(1 - 2x\cos\theta + x^2)^{-1}$ i.e. in
$$1 + x(2\cos\theta - x) + \ldots + x^m(2\cos\theta - x)^m + x^{m+1}(2\cos\theta - x)^{m+1} + \ldots$$

A. If now n is *odd*, we may take $m = \frac{1}{2}(n-1)$; and the coeff. of x^{n-1} will be
$$(-1)^{\frac{n-1}{2}}\left[1 - \frac{\frac{1}{2}(n+1)\cdot\frac{1}{2}(n-1)}{\lfloor 2}(2\cos\theta)^2 + \frac{\frac{1}{2}(n+3)\cdot\frac{1}{2}(n+1)\,\frac{1}{2}(n-1)\,\frac{1}{2}(n-3)}{\lfloor 4}(2\cos\theta)^4 - \ldots\right]$$
$$\therefore \text{ if } n \text{ is } odd,\ (-1)^{\frac{n-1}{2}}\frac{\sin n\theta}{\sin\theta}$$
$$= 1 - \frac{n^2-1^2}{\lfloor 2}\cos^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{\lfloor 4}\cos^4\theta - \ldots \text{ up to } \cos^{n-1}\theta.$$

B. If now n is *even*, we may take $m = \frac{1}{2}n$, and the coeff. of x^{n-1} will be
$$(-1)^{\frac{n-2}{2}}\left[\tfrac{1}{2}n \cdot 2\cos\theta - \frac{\frac{1}{2}(n+2)\cdot\frac{1}{2}n\cdot\frac{1}{2}(n-2)}{\lfloor 3}(2\cos\theta)^3 + \ldots\right]$$
$$\therefore \text{ if } n \text{ is } even,\ (-1)^{\frac{n-2}{2}}\frac{\sin n\theta}{\sin\theta}$$
$$= n\cos\theta - \frac{n(n^2-2^2)}{\lfloor 3}\cos^3\theta + \frac{n(n^2-2^2)(n^2-4^2)}{\lfloor 5}\cos^5\theta - \ldots$$
up to $\cos^{n-1}\theta$.

374. By Art. 371, we have

$2\cos n\theta = \text{coeff. of } x^n - \text{coeff. of } x^{n-2} \text{ in } (1-2x\cos\theta+x^2)^{-1}$, i.e. in

$1 + x(2\cos\theta - x) + \ldots + x^m(2\cos\theta - x)^m + x^{m+1}(2\cos\theta - x)^{m+1} + \ldots.$

C. If now n is *odd*, we may take $m = \frac{1}{2}(n+1)$,

and the coeff. of x^n will be

$$(-1)^{\frac{n-1}{2}}\left[\tfrac{1}{2}(n+1).\,2\cos\theta - \frac{\tfrac{1}{2}(n+3).\tfrac{1}{2}(n+1).\tfrac{1}{2}(n-1)}{\lfloor 3}(2\cos\theta)^3 + \ldots\right]$$

and the coeff. of x^{n-2} will be

$$(-1)^{\frac{n-3}{2}}\left[\tfrac{1}{2}(n-1).\,2\cos\theta - \frac{\tfrac{1}{2}(n+1).\tfrac{1}{2}(n-1)\,\tfrac{1}{2}.(n-3)}{\lfloor 3}(2\cos\theta)^3 + \ldots\right]$$

$$\therefore (-1)^{\frac{n-1}{2}}\cos n\theta \text{ (if } n \text{ is odd)}$$

$$= n\cos\theta - \frac{n(n^2-1^2)}{\lfloor 3}\cos^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{\lfloor 5}\cos^5\theta - \ldots.$$

D. If n is *even*, we may take $m = \frac{1}{2}n$,

and the coeff. of x^n will be

$$(-1)^{\frac{n}{2}}\left[1 - \frac{\tfrac{1}{2}(n+2).\tfrac{1}{2}n}{\lfloor 2}.(2\cos\theta)^2 + \frac{\tfrac{1}{2}(n+4).\tfrac{1}{2}(n+2).\tfrac{1}{2}n.\tfrac{1}{2}(n-2)}{\lfloor 4}(2\cos\theta)^4 - \ldots\right]$$

and the coeff. of x^{n-2} will be

$$(-1)^{\frac{n-2}{2}}\left[1 - \frac{\tfrac{1}{2}n.\tfrac{1}{2}(n-2)}{\lfloor 2}(2\cos\theta)^2 + \frac{\tfrac{1}{2}(n+2).\tfrac{1}{2}n.\tfrac{1}{2}(n-2).\tfrac{1}{2}(n-4)}{\lfloor 4}(2\cos\theta)^4 - \ldots\right]$$

$$\therefore (-1)^{\frac{n}{2}}\cos n\theta \text{ (if } n \text{ is even)}$$

$$= 1 - \frac{n^2}{\lfloor 2}\cos^2\theta + \frac{n^2(n^2-2^2)}{\lfloor 4}\cos^4\theta - \frac{n^2(n^2-2^2)(n^2-4^2)}{\lfloor 6}\cos^6\theta - \ldots.$$

375. In the results of the last articles put $\theta = \frac{1}{2}\pi - \phi$, so that $\sin\theta = \cos\phi$ and $\cos\theta = \sin\phi$. Then convert $\sin n\theta$ and $\cos n\theta$ according as n is odd or even by Ex. XII. 39. Thus

By Art. 370,

$$(n \text{ odd})\,(-1)^{\frac{n-1}{2}}\frac{\cos n\phi}{\cos\phi} = (2\sin\phi)^{n-1} - (n-2)(2\sin\phi)^{n-3} + \ldots,$$

$$(n \text{ even})\,(-1)^{\frac{n-2}{2}}\frac{\sin n\phi}{\cos\phi} = (2\sin\phi)^{n-1} - (n-2)(2\sin\phi)^{n-3} + \ldots$$

By Art. 371,

$$(n \text{ odd})\,(-1)^{\frac{n-1}{2}}.\ 2\sin n\phi = (2\sin\phi)^n - n(2\sin\phi)^{n-2} + \ldots + (-1)^r n.\frac{(n-r-1)\ldots(n-2r+1)}{\lfloor r}(2\sin\phi)^{n-2r} + \ldots,$$

$$(n \text{ even})\,(-1)^{\frac{n}{2}}\,2\cos n\phi = (2\sin\phi)^n - n(2\sin\phi)^{n-2} + \ldots + (-1)^r n.\frac{(n-r-1)\ldots(n-2r+1)}{\lfloor r}(2\sin\phi)^{n-2r} + \ldots.$$

By Art. 373,

$$(n \text{ odd})\,\frac{\cos n\phi}{\cos\phi} = 1 - \frac{n^2-1^2}{\lfloor 2}\sin^2\phi + \frac{(n^2-1^2)(n^2-3^2)}{\lfloor 4}\sin^4\phi - \ldots,$$

$$(n \text{ even})\,\frac{\sin n\phi}{\cos\phi} = n\sin\phi - \frac{n(n^2-2^2)}{\lfloor 3}\sin^3\phi + \frac{n(n^2-2^2)(n^2-4^2)}{\lfloor 5}\sin^5\phi - \ldots.$$

By Art. 374;

$$(n \text{ odd})\,\sin n\phi = n\sin\phi - \frac{n(n^2-1^2)}{\lfloor 3}\sin^3\phi + \frac{n(n^2-1^2)(n^2-3^2)}{\lfloor 5}\sin^5\phi - \ldots,$$

$$(n \text{ even})\,\cos n\phi = 1 - \frac{n^2}{\lfloor 2}\sin^2\phi + \frac{n^2(n^2-2^2)}{\lfloor 4}\sin^4\phi - \frac{n^2(n^2-2^2)(n^2-4^2)}{\lfloor 6}\sin^6\phi + \ldots.$$

376. The results of the last article might be obtained more directly from the two series

$$S = x\cos\phi + x^2\sin 2\phi - x^3\cos 3\phi - x^4\sin 4\phi + \ldots,$$

which gives $S(1 - 2x\sin\phi + x^2) = x\cos\phi,$

and $S = 1 + 2x\sin\phi - 2x^2\cos 2\phi - 2x^3\sin 3\phi + \ldots,$

which gives $S(1 - 2x\sin\phi + x^2) = 1 - x^2.$

377. In the above articles, we have incidentally proved the important expansions:—

$$\frac{1}{1 - 2x\cos\theta + x^2} = 1 + x\cdot\frac{\sin 2\theta}{\sin\theta} + x^2\cdot\frac{\sin 3\theta}{\sin\theta} + \ldots + x^{n-1}\cdot\frac{\sin n\theta}{\sin\theta} + \ldots$$

$$\frac{1 - x^2}{1 - 2x\cos\theta + x^2} = 1 + 2x\cos\theta + 2x^2\cos 2\theta + \ldots + 2x^n\cos n\theta + \ldots.$$

In the first of these expansions, put

$$x = \frac{y\sqrt{c}}{\sqrt{a}} \text{ and } \cos\theta = \frac{b}{\sqrt{(ac)}};$$

then $a - 2by + cy^2 = a(1 - 2x\cos\theta + x^2).$

Thus, if $b^2 < ac$, we have

$$\frac{a}{a - 2by + cy^2} = 1 + \frac{y\sqrt{c}}{\sqrt{a}}\cdot\frac{\sin 2\theta}{\sin\theta} + \ldots + \left(\frac{y\sqrt{c}}{\sqrt{a}}\right)^{n-1}\cdot\frac{\sin n\theta}{\sin\theta} + \ldots$$

where $\theta = \cos^{-1}\{b \div \sqrt{(ac)}\}.$

EXAMPLES XIV.

Prove

1. $\sec 2A = \dfrac{\sec^2 A}{2 - \sec^2 A}$.
2. $\tan 2A = \dfrac{2 \cot A}{\cot^2 A - 1}$.
3. $\operatorname{cosec} 2A = \frac{1}{2} \sec A \operatorname{cosec} A$.
4. $\cot 2A = \frac{1}{2}(\cot A - \tan A)$.
5. $\tan 3A = \dfrac{3 - \tan^2 A}{\cot A - 3 \tan A}$.
6. $\cot 3A = \dfrac{\cot^3 A - 3 \cot A}{3 \cot^2 A - 1}$.
7. $\cos^2 A + \sin^2 A \cos 2B = \cos^2 B + \sin^2 B \cos 2A$.
8. $\sin^2 A - \cos^2 A \cos 2B = \sin^2 B - \cos^2 B \cos 2A$.
9. $\sin x (2 \cos x - 1) = 2 \sin \frac{1}{2}x \cos \frac{1}{2} . 3x$.
10. $\sin x (2 \cos x + 1) = 2 \cos \frac{1}{2}x \sin \frac{1}{2} . 3x$.
11. $\dfrac{\cos(x - 3y) - \cos(3x - y)}{\sin 2x + \sin 2y} = 2 \sin(x - y)$.
12. $\dfrac{\sin(x + 3y) + \sin(3x + y)}{\sin 2x + \sin 2y} = 2 \cos(x + y)$.
13. $\cos 3A = (2 \cos 2A - 1) \cos A$.
14. $\sin 3A = (2 \cos 2A + 1) \sin A$.
15. $\sin 3A \operatorname{cosec} A - \cos 3A \sec A = 2$.
16. $\tan 4A = \dfrac{\sin A \sin 2A + \sin 2A \sin 5A}{\sin A \cos 2A + \sin 2A \cos 5A}$.
17. $\cos A + \cos 3A + \cos 7A + \cos 9A = 4 \cos A \cos 3A \cos 5A$.
18. $2 \cos \theta - \dfrac{1}{2 \cos \theta} - \dfrac{\tan \theta}{2 \sin 3\theta} = \dfrac{\sin 4\theta}{\sin 3\theta}$.
19. $\cos 5\theta = \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta$.
20. $\sin 5\theta = 5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta$.
21. $\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta$.
22. $\sin 5\theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$.
23. $\sin 6\theta \div \sin \theta = 32 \cos^5 \theta - 32 \cos^3 \theta + 6 \cos \theta$.

24. $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$

25. Find $\cos 4A$ and $\sin 4A$ in terms of $\cos A$ and $\sin A$ by means of the formulæ which give $\cos 2A$ and $\sin 2A$ in terms of $\cos A$ and $\sin A$. Thence find $\cos 4A$ and $\sin 4A \div \sin A$ in terms of $\cos A$.

26. Show that the numerator and denominator of $\cot nA$ are, respectively, $K_0 - K_2 + K_4 - \ldots$ and $K_1 - K_3 + K_5 - \ldots$, where K_r = term involving $\cot^{n-r} A$ in the expansion of $(\cot A + 1)^n$.

27. $4\cos A\cos 2A\cos 3A = 1 + \cos 2A + \cos 4A + \cos 6A.$

28. $4\cos(n+1)A\cos nA\cos(n-1)A = \cos 3nA + \cos(n-2)A + \cos nA + \cos(n+2)A.$

29. $2\sin A \,.\, 2\sin B \,.\, 2\sin C \ldots$ to n factors

$$= (-1)^{\frac{n-1}{2}} \Sigma (-1)^r \sin(\pm A \pm B \pm C \pm \ldots)$$

or
$$= (-1)^{\frac{n}{2}} \Sigma (-1)^r \cos(\pm A \pm B \pm C \pm \ldots),$$

according as n is odd or even, where r = the number of negatives in the compound angle which enters into any term considered.

30. From the last example deduce the results of Art. 363.

31. $1 + \cos 9\theta = (1 + \cos\theta)(16\cos^4\theta - 8\cos^3\theta - 12\cos^2\theta + 4\cos\theta + 1)^2.$

32. $(\sin A - \cos A)^4 + (\sin A + \cos A)^4 = 3 - \cos 4A.$

33. $\sin 3A = 4\sin A\sin(60° + A)\sin(60° - A).$

34. $3\operatorname{cosec} 3A = \operatorname{cosec} A + \operatorname{cosec}(A + 120°) + \operatorname{cosec}(A + 240°).$

35. If $\sin B$ be the Geometric mean of $\sin A$ and $\cos A$,
$$\cos 2B = 2\cos^2(A + 45°).$$

36. If $\sin B$ be the Arithmetic mean of $\sin A$ and $\cos A$,
$$\cos 2B = \cos^2(A + 45°).$$

37. $\dfrac{\sin A - \cos A}{\sin A + \cos A} = \tan(A - 45°).$

38. $\dfrac{\sin 5\theta - \cos 5\theta}{\sin 5\theta + \cos 5\theta} = \tan\left(\theta - \dfrac{\pi}{4}\right) \cdot \dfrac{1 - 2\sin 2\theta - 4\sin^2 2\theta}{1 + 2\sin 2\theta - 4\sin^2 2\theta}.$

39. If $\cos(\alpha+\theta)+m\cos\theta=n$, then n^2 is not greater than $1+2m\cos\alpha+m^2$.

40. If $\sqrt{2}\,.\cos A=\cos B+\cos^2 B$,

and $\sqrt{2}\,.\sin A=\sin B-\sin^2 B$,

then $\pm\sin(B-A)=\cos 2B=\frac{1}{3}$.

41. $\tan A\,.\,(\tan 2A)^{\frac{1}{2}}\,.\,(\tan 2^2A)^{\frac{1}{2^2}}$ ... to $n+1$ factors

$$=\frac{4\sin^2 A}{(2\sin 2^{n+1}A)^{1+2^n}}.$$

42. If $\sin(\beta-\gamma)\cos(\theta-2\alpha)\sec\alpha+\sin(\gamma-\alpha)\cos(\theta-2\beta)\sec\beta$
$+\sin(\alpha-\beta)\cos(\theta-2\gamma)\sec\gamma=\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)$,

then $\cos\theta=\cos\alpha\cos\beta\cos\gamma$.

43. If $\dfrac{\sin r\alpha}{l}=\dfrac{\sin(r+1)\alpha}{m}=\dfrac{\sin(r+2)\alpha}{n}$,

then $\dfrac{\cos r\alpha}{2m^2-l(l+n)}=\dfrac{\cos(r+1)\alpha}{m(n-l)}=\dfrac{\cos(r+2)\alpha}{n(l+n)-2m^2}$.

44. If $4\cos(x-y)\cos(y-z)\cos(z-x)=1$,

then $1+12\cos 2(x-y)\cos 2(y-z)\cos 2(z-x)$
$=4\cos 3(x-y)\cos 3(y-z)\cos 3(z-x)$.

45. If $\sin(\theta+\alpha)=\sin(\phi+\alpha)=\sin\beta$,

and $a\sin(\theta+\phi)+b\sin(\theta-\phi)=c$,

then *either* $a\sin(2\alpha\pm 2\beta)=-c$,

or $a\sin 2\alpha\pm b\sin 2\beta=c$.

46. If $A+B+C=180°$, then

$\cos A\sin 3A+\cos B\sin 3B+\cos C\sin 3C$
$=(\sin 2A+\sin 2B+\sin 2C)\,(\frac{3}{2}+\cos 2A+\cos 2B+\cos 2C)$.

47. If $\phi=\frac{1}{2}(\alpha+\beta+\gamma-3\theta)$, then

$\sin^3(\theta-\alpha)\cos 2(\alpha-\phi)\sin(\beta-\gamma)+\sin^3(\theta-\beta)\cos 2(\beta-\phi)\sin(\gamma-\alpha)$
$+\sin^3(\theta-\gamma)\cos 2(\gamma-\phi)\sin(\alpha-\beta)=$
$\frac{1}{2}(\sin 2\alpha+\sin 2\beta+\sin 2\gamma-3\sin 2\theta)\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)$.

Solve the following equations :—

48. $5\tan^2 x - \sec^2 x = 11$.

49. $5\tan^2 x + \sec^2 x = 7$.

50. $\sin 5\theta - \sin 3\theta = \sqrt{2} \cdot \cos 4\theta$.

51. $\cos 5\theta + \cos 3\theta = \sqrt{2} \cdot \cos 4\theta$.

52. $4 \sin\theta \cos 2\theta = 1$.

53. $\sin \frac{1}{2}A = \cos \frac{1}{2} \cdot 5A$.

54. $\tan A + \sec 2A = 1$.

55. $\sec 4\theta - \sec 2\theta = 2$.

56. $\sin 2\theta + \sin 4\theta = \sqrt{2} \cdot \cos\theta$.

57. $\cos 2\theta - \cos 4\theta = \sqrt{2} \cdot \sin\theta$.

58. $\sin\theta + \sin 2\theta + \sin 3\theta = 0$.

59. $\sin\theta + \sin 5\theta = \sin 7\theta$.

60. $\cos 2\theta + \cos 4\theta + \cos\theta = 0$.

61. $\cos 2\theta + \cos 4\theta = \cos\theta$.

62. $\sin 4x = \tan x$.

63. $16 \sin^5\theta = \sin 5\theta - 5 \sin 3\theta$.

64. $\sec^2 3\theta + \operatorname{cosec}^2 3\theta = 4$.

65. $\cot^2 \frac{1}{2}\theta - \sec^2 \frac{1}{2}\theta = 2\sqrt{3} \cot^2\theta$.

66. $2(\sin\alpha - \cos\alpha) + 2m^3(\tan\alpha + \cot\alpha) + \cot\alpha - \sec\alpha = 0$.

67. $8\cos^4\theta + 8\sin^4\theta = 6 + \sin\theta + \sin 9\theta$.

68. $\tan^{-1} x + \tan^{-1}(1 - x) = 2\tan^{-1}\sqrt{(x - x^2)}$.

69. $\sin^{-1} x + \sin^{-1}(1 - x) = \cos^{-1} x$.

70. $x = 3\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2\,6} - \frac{1}{4}\pi$.

CHAPTER XV.

SUBMULTIPLE ANGLES.

378. In the last chapter we found the ratios of nA in terms of those of A: or, which is the same thing, the ratios of A in terms of those of $\frac{A}{n}$.

In this chapter we shall try to find the ratios of $\frac{A}{n}$ in terms of those of A: or, which is the same thing, the ratios of A in terms of those of nA.

To find the number *of solutions.*

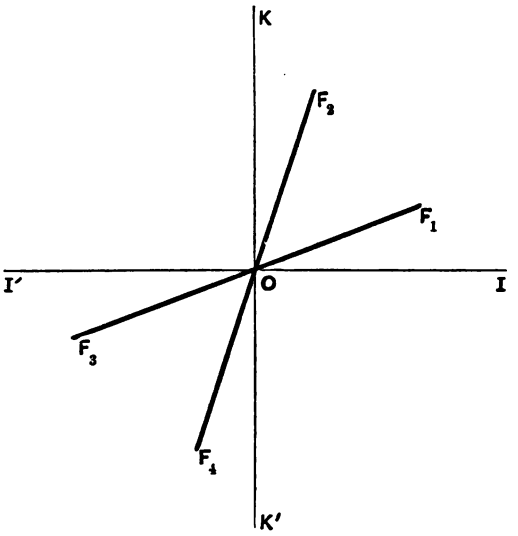
379. *To prove that, in determining the ratios of* $\frac{1}{2}A$, *when nothing is known of A but the value of its* sine [*or* cosecant], *there will be* two reciprocal values *of the tangent and co-tangent and* two pairs of equal and opposite values *of the other ratios.*

Let A_0 be *one* of the values of A for which the sine or cosecant has the given value.

Then [Art. 312, II.] the general value of A is $n.180° + (-1)^n A_0$

$\therefore$ the general value of $\frac{1}{2}A$ is $n.90° + (-1)^n.\frac{1}{2}A_0$.

Give to n in succession the values 0, 1, 2, 3 &c.



Then the final positions of the line describing $\frac{1}{2}A$ will be OF_1, OF_2, OF_3, OF_4, where (see figure) each of the angles IOF_1, F_2OK, $I'OF_3$, F_4OK' is equal to $\frac{1}{2}A_0$.

But the *sine* and *cosine* of IOF_1 are *equal and opposite* to the *sine* and *cosine* of IOF_3 respectively.

And the *sine* and *cosine* of IOF_2 are *equal and opposite* to the *sine* and *cosine* of IOF_4 respectively.

And $\tan IOF_1 = \cot IOF_2 = \tan IOF_3 = \cot IOF_4$.

And $\tan IOF_2 = \cot IOF_1 = \tan IOF_4 = \cot IOF_3$.

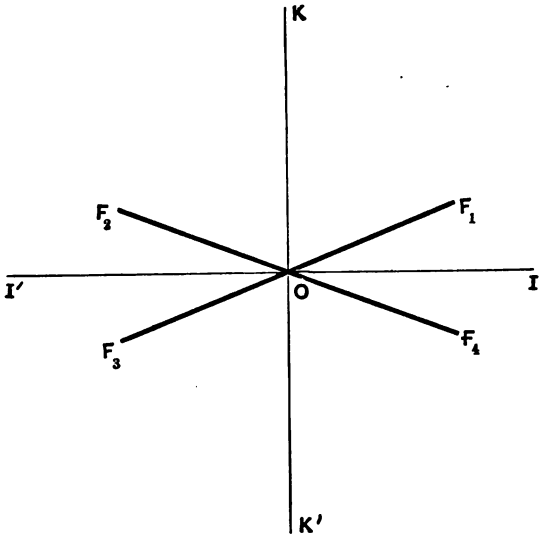
Hence the tangents of IOF_1 and IOF_2 are *reciprocal;* and so on.

Example 1. If $\pm s$, $\pm s'$ are the four values of $\sin\frac{1}{2}A$ corresponding to any given value of $\sin A$, show that $\pm s' = \pm\sqrt{(1 - s^2)}$.

Example 2. If $\sin A = \pm 1$, show that the *four* values of $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$ reduce to *two*, and the *two* values of $\tan\frac{1}{2}A$ reduce to *one*.

Example 3. If $\sin A = 0$, show that the *four* values of $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$ reduce to *three*.

380. *To prove that, in determining the ratios of* $\frac{1}{2}A$, *when nothing is known of* A *but the value of its* **cosine** [or **secant**], *there will be* a pair of equal and opposite values *of each of the ratios.*



As before [Art. 312, IV.] the general value of A is $2n \cdot 180° \pm A_0$,
$\therefore$ the general value of $\frac{1}{2}A$ is $n \cdot 180° \pm \frac{1}{2}A_0$.

Then the final positions of the line describing $\frac{1}{2}A$ will be OF_1, OF_2, OF_3, OF_4, where (see figure) each of the angles IOF_1, F_2OI', $I'OF_3$, F_4OI is equal to $\frac{1}{2}A_0$.

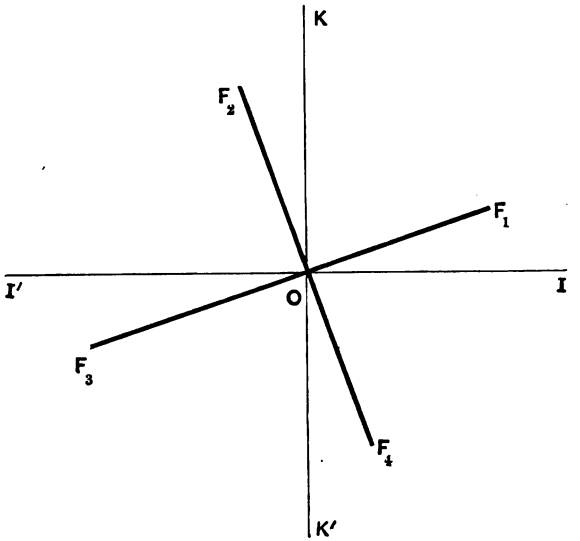
Now $\sin IOF_1 = \sin IOF_2 = -\sin IOF_3 = -\sin IOF_4$,
$\tan IOF_1 = -\tan IOF_2 = \tan IOF_3 = -\tan IOF_4$,
$\sec IOF_1 = -\sec IOF_2 = -\sec IOF_3 = \sec IOF_4$.

Thus there is *a pair of equal and opposite values* of *each* of the ratios of $\frac{1}{2}A$.

Example 1. If $\cos A = 0$, show that there is *still* a pair of equal and opposite values of *each* of the ratios of $\frac{1}{2}A$.

Example 2. If $\cos A = +1$, $\sin \frac{1}{2}A$ has only *one* value. If $\cos A = -1$, $\cos \frac{1}{2}A$ has only *one* value.

381. *To prove that, in determining the ratios of* $\frac{1}{2}A$, *when nothing is known of* A *but the value of its* **tangent** [or **cotangent**], *there will be* two reciprocal but opposite values *of the tangent and cotangent, and* two pairs of equal and opposite values *of the other ratios.*



Here [Art. 312, III.] the general value of A is $n \cdot 180° + A_0$,
$\therefore$ the general value of $\frac{1}{2}A$ is $n \cdot 90° + \frac{1}{2}A_0$.

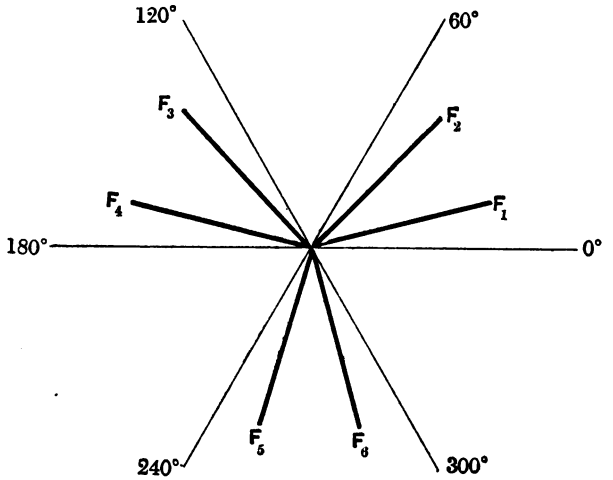
Then the final positions of the line describing $\frac{1}{2}A$ will be OF_1, OF_2, OF_3, OF_4, where (see figure) each of the angles IOF_1, KOF_2, $I'OF_3$, $K'OF_4$ is equal to $\frac{1}{2}A_0$.

Now $\tan IOF_1 = -\cot IOF_2 = \tan IOF_3 = -\cot IOF_4$,
$\cot IOF_1 = -\tan IOF_2 = \cot IOF_3 = -\tan IOF_4$,
$\sin IOF_1 = -\sin IOF_3$: $\sin IOF_2 = -\sin IOF_4$,
$\cos IOF_1 = -\cos IOF_3$: $\cos IOF_2 = -\cos IOF_4$.

Example 1. If $\pm s$, $\pm s'$ are the four values of $\sin \frac{1}{2}A$, corresponding to any given value of $\tan A$, show that $s^2 + s'^2 = 1$.

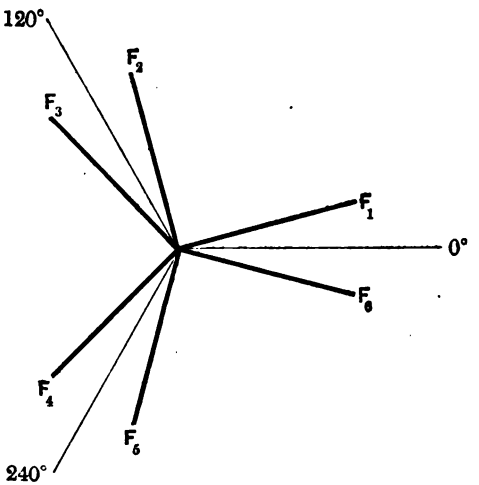
Example 2. If $\tan A = 0$, show that $\sin \frac{1}{2}A$ and $\cos \frac{1}{2}A$ have each *three* values. How many values have the other ratios of $\frac{1}{2}A$?

382. See below.



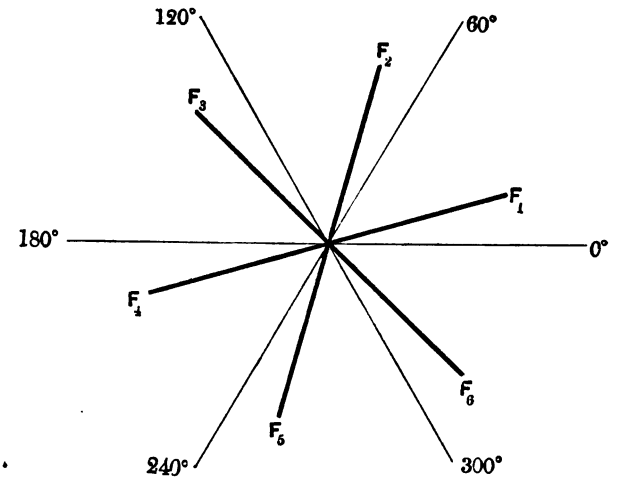
Here: Given $\sin A$; $A = n \cdot 180° + (-1)^n A_0$,

$\therefore \frac{1}{3}A = n \cdot 60° + (-1)^n \frac{1}{3}A_0$.



Here: Given $\cos A$; $A = n \cdot 360° \pm A_0$,

$\therefore \frac{1}{3}A = n \cdot 120° \pm \frac{1}{3}A_0$.



Here: Given $\tan A$; $A = n \cdot 180^0 + A_0$,

$$\therefore \tfrac{1}{3}A = n \cdot 60^0 + \tfrac{1}{3}A_0.$$

In determining the ratios of $\frac{1}{3}A$, *when we know nothing but the value of* one *of the ratios of* A, *there will be* three *values of the* corresponding *ratio of* $\frac{1}{3}A$ (and of its reciprocal), *but* three pairs of equal and opposite values *of the other ratios of* $\frac{1}{3}A$.

The accompanying figures will show this.

383. In Arts. 384, 385, 386 we shall find the number of values of the ratios of $\frac{A}{n}$ when a ratio of A is given.

We shall write π for 180^0.

It should be noted that for some *special* values of the given ratio of A, the number of values of any ratio of $\frac{A}{n}$ is *less* than that given.

Hence the results must be understood to apply to any *general* value of the given ratio.

This was noted in the examples given above.

384. *Given* sin A, *to find the number of values of each of the ratios of* $\frac{A}{n}$.

Here, the general value of A is $\lambda\pi + (-1)^{\lambda} . \alpha$.

$\therefore$ the general value of $\frac{A}{n}$ is $\frac{\lambda\pi}{n} + (-1)^{\lambda} . \frac{\alpha}{n}$.

The two values of *any* ratio of A/n, obtained by giving to λ different values μ and μ', will be equal *if the two values* of A/n *differ by an* even *multiple of* π.

i.e. if $$\left\{\frac{\mu\pi}{n} + (-1)^{\mu} . \frac{\alpha}{n}\right\} \sim \left\{\frac{\mu'\pi}{n} + (-1)^{\mu'} . \frac{\alpha}{n}\right\}$$

is an even multiple of π.

i.e. if * $\mu \sim \mu'$ = an *even* multiple of n.

Hence the values of any ratio of A/n, obtained by giving to λ in succession the $2n$ values 0, 1, 2...$(2n-1)$, will be *repeated*, if any *other* integral value is given to λ.

Hence no ratio of A/n can have more than the $2n$ values so obtained.

Now consider each of the ratios sin A/n, cos A/n, and tan A/n separately.

(1) *Consider* sin A/n.

Two values of sin A/n will be equal, if *the sum of the two values of* A/n *is an* odd *multiple of* π.

i.e. if $$\frac{\mu\pi}{n} + (-1)^{\mu} . \frac{\alpha}{n} + \frac{\mu'\pi}{n} + (-1)^{\mu'} . \frac{\alpha}{n}$$

is an odd multiple of π.

i.e. if $\mu + \mu'$ is *odd*, and an *odd* multiple of n.

This cannot be unless n is *odd*.

If n is *odd*, the values of sin A/n, obtained by giving to λ in succession the n *even* values, 0, 2...$n-1$; $n+1$,...$2n-2$ will be repeated by giving the n *odd* values n, $n-2$,...1; $2n-1$,...$n+2$.

* For, since $\mu \sim \mu'$ is even, μ and μ' must be either *both even* or *both odd*. Hence the terms involving α cancel.

Thus, given $\sin A$,

if n is even, there are $2n$ values of $\sin A/n$;

if n is odd, there are n values of $\sin A/n$.

(2) *Consider* $\cos A/n$.

Two values of $\cos A/n$ will be equal, if *the sum of the two values of* A/n *is an* even *multiple of* π.

i.e. if $\mu + \mu'$ is *odd*, but an *even* multiple of n.

This is a contradiction.

Thus, given $\sin A$,

there are always $2n$ values of $\cos A/n$.

(3) *Consider* $\tan A/n$.

Two values of $\tan A/n$ will be equal if *the difference of the two values of* A/n *is an* odd *multiple of* π.

i.e. if $\mu \sim \mu'$ is *even*, and an *odd* multiple of n.

This cannot be unless n is *even*.

If n is *even*, the values of $\tan A/n$, obtained by giving to λ in succession the n values, $0, 1, 2 \ldots n-1$, will be repeated by giving the n values, $n, n+1, n+2, \ldots 2n-1$.

Thus, given $\sin A$,

if n is odd, there are $2n$ values of $\tan A/n$;

if n is even, there are n values of $\tan A/n$.

385. *Given* $\cos A$, *to find the number of values of each of the ratios of* $\frac{A}{n}$.

Here, the general value of A is $2\lambda\pi \pm a$,

$\therefore$ the general value of $\frac{A}{n}$ is $\frac{2\lambda\pi}{n} \pm \frac{a}{n}$.

The two values of *any* ratio of A/n, obtained by giving to λ different values μ and μ', will be equal *if the two values of* A/n *differ by an* even *multiple of* π.

i.e. if $$\left(\frac{2\mu\pi}{n} \pm \frac{a}{n}\right) \sim \left(\frac{2\mu'\pi}{n} \pm \frac{a}{n}\right)$$

is an even multiple of π.

i.e. if, taking the same sign before α/n, $\mu \sim \mu'$ is a multiple of n.

Hence the values of any ratio of A/n, obtained by giving to λ in succession the n values 0, 1, 2...$n-1$ and either sign before α/n, will be *repeated*, if any *other* integral value is given to λ with the *same* sign before α/n.

Hence no ratio of A/n can have more than the $2n$ values so obtained.

(1) *Consider* $\sin A/n$.

Two values of $\sin A/n$ will be equal

if $$\frac{2\mu\pi}{n} \pm \frac{\alpha}{n} + \frac{2\mu'\pi}{n} \pm \frac{\alpha}{n}$$ is an *odd* multiple of π.

i.e. if, taking opposite signs before α/n, $2(\mu+\mu')$ is an *odd* multiple of n.

This cannot be unless n is even.

If n is even, the values of $\sin A/n$, obtained by giving to λ in succession the n values 0, 1,...$\frac{1}{2}n$; $\frac{1}{2}n+1$,...$n-1$ with + sign before α/n, will be repeated by giving the n values $\frac{1}{2}n$, $\frac{1}{2}n-1$,... 0; $n-1$,...$\frac{1}{2}n+1$ with − sign before α/n.

Thus, given $\cos A$,

if n is odd, there are $2n$ values of $\sin A/n$,
if n is even, there are n values of $\sin A/n$.

(2) *Consider* $\cos A/n$.

Two values of $\cos A/n$ will be equal, if the sum of the two values of A/n is an *even* multiple of π.

i.e. if, taking opposite signs before α/n, $\mu+\mu'$ is a multiple of n.

Thus the values of $\cos A/n$, obtained by giving to λ the n values 0, 1, 2,...$n-1$ with + sign before α/n, will be repeated by giving the n values 0, $n-1$, $n-2$,...1 with − sign before α/n.

Thus, given $\cos A$,

there are always n values of $\cos A/n$.

(3) *Consider* $\tan A/n$.

Two values of $\tan A/n$ will be equal, if the difference of the two values of A/n is an *odd* multiple of π.

i.e. if, taking the same signs before α/n, $2(\mu \sim \mu')$ is an *odd* multiple of n.

This cannot be unless n is even.

If n is even, the values of $\tan A/n$, obtained by giving to λ the $\frac{1}{2}n$ values $0, 1, 2 \ldots \frac{1}{2}n-1$ with either sign before α/n will be repeated by giving the $\frac{1}{2}n$ values $\frac{1}{2}n, \frac{1}{2}n+1, \frac{1}{2}n+2, \ldots n-1$ with the same sign before α/n.

Thus, given $\cos A$,

if n is odd, there are $2n$ values of $\tan A/n$,

if n is even, there are n values of $\tan A/n$.

386. *Given* $\tan A$, *to find the number of values of each of the ratios of* $\frac{A}{n}$.

Here, the general value of A is $\lambda\pi + \alpha$,

$\therefore$ the general value of $\frac{A}{n}$ is $\frac{\lambda\pi}{n} + \frac{\alpha}{n}$.

$\therefore$ the two values of *any* ratio of A/n will be equal,

if $\left(\frac{\mu\pi}{n} + \frac{\alpha}{n}\right) \sim \left(\frac{\mu'\pi}{n} + \frac{\alpha}{n}\right)$ is an *even* multiple of π.

i.e. if $\mu \sim \mu'$ is an even multiple of n.

Hence, given $\tan A$, there cannot be more than the $2n$ values of any ratio of A/n, obtained by giving to λ the $2n$ values $0, 1, 2 \ldots 2n-1$.

(1) *Consider* $\sin A/n$ *and* $\cos A/n$.

Two values of $\sin A/n$ or $\cos A/n$ will be equal

if $\frac{\mu\pi}{n} + \frac{\alpha}{n} + \frac{\mu'\pi}{n} + \frac{\alpha}{n}$ is a multiple of π.

This cannot be.

Hence, given $\tan A$,

there are always $2n$ values of $\sin A/n$ and of $\cos A/n$.

(2) *Consider* $\tan A/n$.

Two values of $\tan A/n$ will be equal

if $\mu \sim \mu'$ is an *odd* multiple of n.

Hence, the values of $\tan A/n$ obtained by giving to λ the n values $0, 1, 2 \ldots n-1$ will be repeated by giving the values $n, n+1, n+2, \ldots 2n-1$.

Hence, given $\tan A$,

there are always n values of $\tan A/n$.

To find the values *of the solutions.*

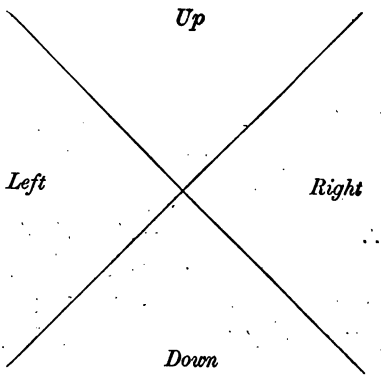
387. *To trace the changes in sign of* $\cos B + \sin B$ *and* $\cos B - \sin B$.

We have

$$\cos B + \sin B = \sqrt{2}\,(\sin 45^\circ \cos B + \cos 45^\circ \sin B) = \sqrt{2} \cdot \sin (45^\circ + B),$$
$$\cos B - \sin B = \sqrt{2}\,(\cos 45^\circ \cos B - \sin 45^\circ \sin B) = \sqrt{2} \cdot \cos (45^\circ + B).$$

Thus

$\cos B + \sin B$ is $+^{ve}$, if $45^\circ + B$ is in 1st or 2nd quadrant,
" " $-^{ve}$, " " 3rd or 4th "
$\cos B - \sin B$ is $+^{ve}$, " " 1st or 4th "
" " $-^{ve}$, " " 2nd or 3rd "



Now B is 45° less than $45° + B$.

Hence, if we turn the initial line from which B is measured *backwards* through 45°,

we shall get four shifted quadrants.

Taking as usual the initial line of B *rightwards*, and the positive revolution *right-upwards*, we have

If B is in the **Right** quadrant, $45° + B$ is in the 1st quadrant,

$\therefore \cos B + \sin B$ is $+^{ve}$; and $\cos B - \sin B$ is $+^{ve}$.

If B is in the **Up** quadrant, $45° + B$ is in the 2nd quadrant,

$\therefore \cos B + \sin B$ is $+^{ve}$; and $\cos B - \sin B$ is $-^{ve}$.

If B is in the **Left** quadrant, $45° + B$ is in the 3rd quadrant,

$\therefore \cos B + \sin B$ is $-^{ve}$; and $\cos B - \sin B$ is $-^{ve}$.

If B is in the **Down** quadrant, $45° + B$ is in the 4th quadrant,

$\therefore \cos B + \sin B$ is $-^{ve}$; and $\cos B - \sin B$ is $+^{ve}$.

388. It is also desirable to notice that

If B is in the *Right* or *Left* quadrant, $2B$ is in the 1st or 4th quadrant.

If B is in the *Up* or *Down* quadrant, $2B$ is in the 2nd or 3rd quadrant.

389. *Given the* **sine** *of an angle to find the ratios of the* half-angle.

We have

$$\cos^2 \tfrac{1}{2}A + \sin^2 \tfrac{1}{2}A = 1 \text{ and } 2 \cos \tfrac{1}{2}A \sin \tfrac{1}{2}A = \sin A.$$

First add and then subtract these equations; thus

$$(\cos \tfrac{1}{2}A + \sin \tfrac{1}{2}A)^2 = 1 + \sin A \text{ and } (\cos \tfrac{1}{2}A - \sin \tfrac{1}{2}A)^2 = 1 - \sin A,$$

$$\therefore \cos \tfrac{1}{2}A + \sin \tfrac{1}{2}A = \pm \sqrt{(1 + \sin A)}$$

and

$$\cos \tfrac{1}{2}A - \sin \tfrac{1}{2}A = \pm \sqrt{(1 - \sin A)}.$$

By Art. 387, the upper or lower sign in these radicals must be chosen according as $\frac{1}{2}A$ is in the Right, Up, Left, or Down quadrant.

Then, by *addition* we get $2\cos\frac{1}{2}A$; by *subtraction*, $2\sin\frac{1}{2}A$.

Thus:

(1) If $\frac{1}{2}A$ is in the *Right* quadrant, the signs are $+\ +$;

$$\therefore\ 2\cos\tfrac{1}{2}A = +\sqrt{(1+\sin A)} + \sqrt{(1-\sin A)},$$
$$2\sin\tfrac{1}{2}A = +\sqrt{(1+\sin A)} - \sqrt{(1-\sin A)}.$$

(2) If $\frac{1}{2}A$ is in the *Up* quadrant, the signs are $+\ -$;

$$\therefore\ 2\cos\tfrac{1}{2}A = +\sqrt{(1+\sin A)} - \sqrt{(1-\sin A)},$$
$$2\sin\tfrac{1}{2}A = +\sqrt{(1+\sin A)} + \sqrt{(1-\sin A)}.$$

(3) If $\frac{1}{2}A$ is in the *Left* quadrant, the signs are $-\ -$;

$$\therefore\ 2\cos\tfrac{1}{2}A = -\sqrt{(1+\sin A)} - \sqrt{(1-\sin A)},$$
$$2\sin\tfrac{1}{2}A = -\sqrt{(1+\sin A)} + \sqrt{(1-\sin A)}.$$

(4) If $\frac{1}{2}A$ is in the *Down* quadrant, the signs are $-\ +$;

$$\therefore\ 2\cos\tfrac{1}{2}A = -\sqrt{(1+\sin A)} + \sqrt{(1-\sin A)},$$
$$2\sin\tfrac{1}{2}A = -\sqrt{(1+\sin A)} - \sqrt{(1-\sin A)}.$$

Now, dividing $2\sin\frac{1}{2}A$ by $2\cos\frac{1}{2}A$, and rationalising the denominators, we have

(5) If $\frac{1}{2}A$ is in the *Right* or *Left* quadrant,

$$\tan\frac{A}{2} = \frac{\sqrt{(1+\sin A)} - \sqrt{(1-\sin A)}}{\sqrt{(1+\sin A)} + \sqrt{(1-\sin A)}} = \frac{1-\sqrt{(1-\sin^2 A)}}{\sin A}.$$

(6) If $\frac{1}{2}A$ is in the *Up* or *Down* quadrant,

$$\tan\frac{A}{2} = \frac{\sqrt{(1+\sin A)} + \sqrt{(1-\sin A)}}{\sqrt{(1+\sin A)} - \sqrt{(1-\sin A)}} = \frac{1+\sqrt{(1-\sin^2 A)}}{\sin A}$$

390. The student should observe that the results of the last article *include* those of Art. 379, which were arrived at by more general considerations.

391. The student should particularly note that

$$\tan\frac{A}{2} = \frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A} = \frac{2\sin^2\frac{1}{2}A}{2\sin\frac{1}{2}A\cos\frac{1}{2}A} = \frac{1-\cos A}{\sin A},$$

for *all* values of A whatever.

But that $\cos A = +\sqrt{(1-\sin^2 A)}$, only if A is in the 1st or 4th quadrant, i.e. if $\frac{1}{2}A$ is in the *Right* or *Left* quadrant. See Art. 388.

And $\cos A = -\sqrt{(1-\sin^2 A)}$, if A is in the 2nd or 3rd quadrant, i.e. if $\frac{1}{2}A$ is in the *Up* or *Down* quadrant. See Art. 388.

These conclusions confirm the value of $\tan \frac{1}{2}A$ as given in Art. 389.

392. *Given the* **cosine** *of an angle, to find the ratios of the* half-angle.

We have

$$\cos^2 \tfrac{1}{2}A + \sin^2 \tfrac{1}{2}A = 1 \text{ and } \cos^2 \tfrac{1}{2}A - \sin^2 \tfrac{1}{2}A = \cos A.$$

First add and then subtract these equations; thus

$$2\cos^2 \tfrac{1}{2}A = 1 + \cos A \text{ and } 2\sin^2 \tfrac{1}{2}A = 1 - \cos A,$$

$$\cos \frac{A}{2} = \pm\sqrt{\left(\frac{1+\cos A}{2}\right)} \text{ and } \sin \frac{A}{2} = \pm\sqrt{\left(\frac{1-\cos A}{2}\right)}.$$

Dividing we have

$$\tan \frac{A}{2} = \pm\sqrt{\left(\frac{1-\cos A}{1+\cos A}\right)}.$$

The signs of these radicals must be taken according to the quadrant in which $\frac{1}{2}A$ lies.

These results include those of Art. 380.

393. *Given the* **tangent** *of an angle, to find the tangent of the* half-angle.

We have

$$\frac{2\tan \frac{1}{2}A}{1-\tan^2 \frac{1}{2}A} = \tan A,$$

$$\therefore \quad \tan A \tan^2 \tfrac{1}{2}A + 2\tan \tfrac{1}{2}A = \tan A,$$

$$\therefore \quad \tan^2 \tfrac{1}{2}A + \frac{2\tan \frac{1}{2}A}{\tan A} + \frac{1}{\tan^2 A} = \frac{\tan^2 A + 1}{\tan^2 A},$$

$$\therefore \quad \tan \frac{A}{2} = \frac{\pm\sqrt{(1+\tan^2 A)} - 1}{\tan A}.$$

The upper or lower sign must be taken according as $\tan \frac{1}{2}A$ and $\tan A$ have or have not the same sign.

i.e. according as A is or is not in the 1st or 4th quadrant:

i.e. according as $\frac{1}{2}A$ is or is not in the *Right or Left* quadrant.

394. As in Art. 391, the student should particularly note that

$$\tan \frac{A}{2} = \frac{1 - \cos A}{\sin A} = \frac{\sec A - 1}{\tan A},$$

for *all* values of A whatever.

But that $\sec A = + \sqrt{(1 + \tan^2 A)}$, only if A is in the 1st or 4th quadrant: while $\sec A = - \sqrt{(1 + \tan^2 A)}$, if A is in the 2nd or 3rd quadrant.

These conclusions confirm the value of $\tan \frac{1}{2}A$ given in the last article.

The cubic equation.

395. The ratios of $\frac{1}{3}A$ can be obtained algebraically from those of A only by the solution of a *cubic* equation.

We must, therefore, give some account of the cubic equation and its connexion with Trigonometrical formulæ.

The general form of the cubic equation is

$$AX^3 + BX^2 + CX + D = 0.$$

If here we write for X, $x - \frac{B}{3A}$, the coefficient of x^2 will vanish.

We may thus reduce any cubic equation to the form

$$x^3 + px + q = 0.$$

It is convenient to express this in the following *homogeneous* form:

$$x^3 \mp 3a^2x = 2b^3,$$

where, a and b being real (i.e. positive or negative), the *upper* or *lower* sign is taken, according as the coefficient of x is arithmetically *positive* or *negative*.

396. *To find the real value of x which satisfies the equation*

$$x^3 - 3a^2x = 2b^3$$

where b is numerically greater than a.

Put $x \equiv y + a^2/y$ (a homogeneous expression),

$\therefore \quad y^3 + 3a^2y + 3a^4/y + a^6/y^3 - 3a^2y - 3a^4/y = 2b^3,$

$\therefore$ (multiplying by y^3)

$$y^6 - 2b^3y^3 + b^6 = b^6 - a^6$$

$$\therefore \quad y^3 = b^3 \pm \sqrt{(b^6 - a^6)}.$$

Since $b^2 > a^2$, y^3 is real; $\therefore$ y has one and only one real value [Art. 230] corresponding to each of the values of y^3.

Now
$$\frac{a^6}{y^3} = \frac{a^6}{b^3 \pm \sqrt{(b^6 - a^6)}} = \frac{a^6\{b^3 \mp \sqrt{(b^6 - a^6)}\}}{b^6 - (b^6 - a^6)}$$
$$= b^3 \mp \sqrt{(b^6 - a^6)}.$$

Substituting for y to find x, we have

$$x = \sqrt[3]{\{b^3 + \sqrt{(b^6 - a^6)}\}} + \sqrt[3]{\{b^3 - \sqrt{(b^6 - a^6)}\}}.$$

397. Similarly to solve

$$x^3 + 3a^2x = 2b^3$$

put $x = y - a^2/y$, and we have

$$x = \sqrt[3]{\{b^3 + \sqrt{(b^6 + a^6)}\}} + \sqrt[3]{\{b^3 - \sqrt{(b^6 + a^6)}\}}.$$

398. By the last two articles, we have found one real solution of the equation

$$x^3 - px + q = 0$$

in the case when $\left(\frac{p}{3}\right)^3$ is algebraically less than $\left(\frac{q}{2}\right)^2$.

Example 1. Solve $x^3 - 18x = 35$.
Example 2. Solve $4x^3 = 5x - 6$.
Example 3. Solve $x^3 = 36x + 91$.

399. *To find three values of x to satisfy the equation*

$$x^3 - 3a^2x = 2b^3$$

where b is numerically $=$ or $< a$.

Put $x \equiv 2a\cos\theta$,

$\therefore \quad 8a^3\cos^3\theta - 6a^3\cos\theta = 2b^3$,

$\therefore \quad 4\cos^3\theta - 3\cos\theta = b^3/a^3$,

i.e. $\quad \cos 3\theta = b^3/a^3$.

Now since b^3 is numerically not greater than a^3, there are real values of 3θ which satisfy this equation.

Let the series of values be 3α, $360° - 3\alpha$, $360° + 3\alpha$, &c.

Then $x \equiv 2a\cos\theta$

$= 2a\cos\alpha$ *or* $2a\cos(120° - \alpha)$ *or* $2a\cos(120° + \alpha)$ &c.

giving three and three only real values of x.

Example. Solve $2y^3 = 3y - 1$.

400. By the last article we have shown how three real roots of the equation $x^3 - px + q = 0$ can be found when $\left(\frac{p}{3}\right)^3$ is algebraically not less than $\left(\frac{q}{2}\right)^2$.

Thus we should find, by means of Trigonometrical tables if necessary, the angle whose cosine $= -\frac{q}{2} \div \sqrt{\left(\frac{p}{3}\right)^3}$.

Then, also with the help of Trigonometrical tables if necessary, we should find the cosine of $\frac{1}{3}$ of this angle.

Finally $x = 2\sqrt{\left(\frac{p}{3}\right)}\cos\tfrac{1}{3}\cos^{-1}\left\{-\sqrt{\left(\frac{27q^2}{4p^3}\right)}\right\}$.

Thus, instead of finding the ratios of $\frac{1}{3}A$ from those of A *by solving a cubic equation*, we do the reverse; i.e. we solve a cubic equation by finding (with the help of tables if necessary) the ratios of $\frac{1}{3}A$ from those of A.

401. A geometrical mode of solving the equation

$$x^3 - px + q = 0$$

thus requires us to *trisect* the angle whose cosine is $-\sqrt{(27q^2 \div 4p^3)}$.

An angle cannot be trisected by *ruler and compass*. But the following explanation indicates how a simple instrument may be used for the trisection of an angle.

Let IOF be the given angle.

Let a rod OR revolve round O, so that R describes a circle which cuts IO produced in I'.

Let MV—rigidly attached to OR—bisect OR at right-angles at M; and suppose it to be *slit* along its central axis.

Also let RI'—pivoted at R to OR—be slit along its central axis, and always pass through a pin fixed at I'.

Thus a pin P, through each of the slits, will mark the point of intersection of MV and RI'.

Then, *if OR rotates until P is on OF, OR shall trisect the angle IOF.*

For $\because\ OR = OI'$, $\therefore\ \angle\ ORI' = \angle\ OI'R$, and exterior angle IOR = *twice* interior angle ORI' = *twice* angle ROP (because $\triangle$'s MOP, MRP are equal in all respects).

$$\therefore\ \angle\ ROP = \tfrac{1}{3}\ \angle\ IOF. \qquad \text{Q.E.D.}$$

Ratios of Particular Angles.

402. We may find the ratios of angles which have a simple relation to a right-angle by means of the general formulæ of this and the preceding chapter, without the special geometrical constructions of Chapter IV. Thus

403. *To find sin* 45°.

Since 45° is its own complement,

$$\sin^2 45° = \cos^2 45°, \text{ i.e. } 1 - \sin^2 45°;$$

$$\therefore\ 2\sin^2 45° = 1,\ \therefore\ \sin 45° = \frac{1}{\sqrt{2}}.$$

404. *To find sin* 30°.

The complement of 30° is 60°, i.e. 2.30°;

$$\therefore\ \cos 30° = \sin 60°, \text{ i.e. } 2\sin 30°\cos 30°.$$

But cos 30° is *not* 0; $\therefore$, dividing by cos 30°,

$$2\sin 30° = 1,\ \therefore\ \sin 30° = \tfrac{1}{2}.$$

405. *To find* $\tan 22\frac{1}{2}°$.

$$\tan 22\tfrac{1}{2}° = \frac{\sqrt{(1+\tan^2 45°)}-1}{\tan 45°} = \sqrt{2}-1.$$

406. *To find* $\tan 15°$.

$$\tan 15° = \frac{\sqrt{(1+\tan^2 30°)}-1}{\tan 30°} = \sqrt{(\cot^2 30°+1)} - \cot 30°$$

$$= \sqrt{(3+1)} - \sqrt{3} = 2-\sqrt{3}.$$

407. *To find* $\sin 18°$.

The complement of 36° (i.e. 2.18°) is 54° (i.e. 3.18°);

$$\therefore \sin 36° = \cos 54°,$$

$$\therefore 2\sin 18°\cos 18° = \cos^3 18° - 3\cos 18°\sin^2 18°.$$

But $\cos 18°$ is *not* 0: $\therefore$, dividing by $\cos 18°$,

$$\therefore 2\sin 18° = \cos^2 18° - 3\sin^2 18°$$

$$= 1 - 4\sin^2 18°;$$

$$\therefore \sin^2 18° + \tfrac{1}{2}\sin 18° + \tfrac{1}{16} = \tfrac{5}{16},$$

$$\therefore \sin 18° = \frac{\sqrt{5}-1}{4}$$

(the positive sign of the radical being taken to make $\sin 18°$ positive).

Examples XV.

1. Draw and explain a figure giving all values of θ which satisfy the equations:

a.	$\sin 2\theta = \sin 60°$.	*b.*	$\cos 2\theta = \sin 60°$.
c.	$\tan 2\theta = \tan 75°$.	*d.*	$\cot 2\theta = \tan 75°$.
e.	$\sec 3\theta = 2$.	*f.*	$\operatorname{cosec} 3\theta = \sqrt{5}+1$.
g.	$\sin 3\theta = \sin 135°$.	*h.*	$\tan 3\theta = -\sqrt{3}$.
k.	$\cos 3\theta = 1$.	*l.*	$\sin 3\theta = 0$.

2. Find the ratios of 9°, $4\frac{1}{2}°$, 6°, $7\frac{1}{2}°$, 3°.

3. Express $\cos A/2^n$ in terms of $\cos A$.

4. Trace the changes in sign and magnitude of $\sin \frac{1}{2}A + \cos \frac{1}{2}A$ and of $\sin \frac{1}{2}A - \cos \frac{1}{2}A$, as A varies from 0 to 360°.

5. Show that $\cos \frac{1}{2}A - \sin \frac{1}{2}A = (-1)^m \sqrt{(1 - \sin A)}$ where m is the integral part of $(A + 270°) \div 360°$.

6. Given $\cot A = k$, show that $\cot \frac{1}{4}A$ or $\tan \frac{1}{4}A$ may be obtained from the equation

$$\cot \tfrac{1}{4}A - \tan \tfrac{1}{4}A = 2k \pm 2\sqrt{(1 + k^2)}.$$

Show how the sign of the radical must be determined: and also the sign of the radical in the expression obtained by solving for $\cot \frac{1}{4}A$.

7. If $\sin 4A = s$, show that the four values of $\tan A$ are given by $\{\sqrt{(1+s)} - 1\}\{\sqrt{(1-s)} + 1\} \div s$.

8. Find when $\sin B$ is arithmetically and when it is algebraically greater or less than $\cos B$. Hence deduce the variations in sign of $\cos B - \sin B$, and $\cos B + \sin B$.

9. Show that the three roots of the equation

$$x(x-3)^2 = 4 \sin^2 a$$

are *positive*, and that the difference between the greatest and least of them is > 3 and $< 2\sqrt{3}$.

10. Given θ, construct ϕ geometrically from the equation

$$\sin \phi \sqrt{\left(1 - \tfrac{3}{4} \sin^2 \frac{\theta}{2}\right)} = \sin \frac{\theta}{2}.$$

11. Show that a root of the equation

$$x^3/7 = 3x - 1$$

is

$$6 \sin (\tfrac{1}{7} \text{ of } 90°) - 1.$$

12. Show that $\cot \frac{1}{2}\theta > 1 + \cot \theta$ for all values of θ from 0 to π.

13. If $\tan \dfrac{\theta}{2} = \dfrac{\tan \theta + m - 1}{\tan \theta + m + 1}$, show that m cannot lie between + 1 and − 1.

14. Show that

$$\sec A = \frac{\cos \frac{1}{2}A}{\sqrt{(1+\sin A)}} + \frac{\sin \frac{1}{2}A}{\sqrt{(1-\sin A)}},$$

and show how the signs of the radicals must be determined.

15. Show that, for any value of a,

(i) $\cos a + \cos(120° - a) + \cos(120° + a) = 0$,

(ii) $\cos a \cos(120° - a) + \cos a \cos(120° + a) + \cos(120° - a)\cos(120° + a) = -\frac{3}{4}$,

(iii) $4 \cos a \cos(120° - a)\cos(120° + a) = \cos 3a$.

What relations between the *roots* and the *coefficients* of a cubic equation follow from the above?

Prove

16. $\tan \frac{A}{2} = \pm \sqrt{\left(\frac{2\sin A - \sin 2A}{2\sin A + \sin 2A}\right)}$.

17. $\sin \frac{2\pi}{7} + \sin \frac{4\pi}{7} - \sin \frac{6\pi}{7} = 4 \sin \frac{\pi}{7} \sin \frac{3\pi}{7} \sin \frac{5\pi}{7}$.

18. $\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} = -1 - 4 \cos \frac{\pi}{7} \cos \frac{3\pi}{7} \cos \frac{5\pi}{7}$.

19. If $A + B + C$ be an odd multiple of π,

$$\cot \tfrac{1}{2}C(1 - \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B) = \tan \tfrac{1}{2}A + \tan \tfrac{1}{2}B.$$

20. If $\tan \frac{1}{2}(B + C - A) \tan \frac{1}{2}(C + A - B) \tan \frac{1}{2}(A + B - C) = 1$,

then $\sin 2A + \sin 2B + \sin 2C = 4 \cos A \cos B \cos C$.

21. If $\tan^2 \phi = \sin \theta \sin \theta' \sec^2 \frac{1}{2}(\theta + \theta')$,

then $\tan^2 \frac{1}{2}\phi = \tan \frac{1}{2}\theta \tan \frac{1}{2}\theta'$.

22. If $\cot \frac{1}{2}\alpha + \cot \frac{1}{2}\beta = 2 \cot \theta$,

then $\{1 - 2 \sec\theta \cos(\alpha - \theta) + \sec^2\theta\}\{1 - 2 \sec\theta \cos(\beta - \theta) + \sec^2\theta\} = \tan^4 \theta$.

23. If

$$\frac{\cos^2 \frac{1}{2}(\alpha - \beta) - \sin^2 \frac{1}{2}(\beta + \gamma)}{\cos^2 \frac{1}{2}\alpha - \sin^2 \frac{1}{2}\gamma} = \frac{\cos^2 \frac{1}{2}(\alpha - \beta) - \sin^2 \frac{1}{2}(\alpha + \gamma)}{\cos^2 \frac{1}{2}\beta - \sin^2 \frac{1}{2}\gamma},$$

then $\sin \alpha + \sin \beta + \sin \gamma = 0$,

provided the sum or difference of no two of the angles α, β, γ is any multiple of π.

24. $\tan^{-1} a - \tan^{-1} z$,

$$= \tan^{-1} \frac{a-b}{1+ab} + \tan^{-1} \frac{b-c}{1+bc} + \&c. + \tan^{-1} \frac{y-z}{1+yz}.$$

25. $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{1}{4}\pi$.

26. $\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} = \frac{1}{4}\pi$.

27. If $4 \cos\theta + 3 \sin\theta = 1$, find θ;

given $\sin 53° 7' 51'' = \cdot 8$, $\sin 11° 31' 10'' = \cdot 2$.

28. If $a \cos\theta + b \sin\theta = c$, show that

$$\theta = \tan^{-1} \frac{b}{a} + \cos^{-1} \frac{c}{\sqrt{(a^2+b^2)}}.$$

29. If $\frac{1}{2 \sin 2\theta} = \frac{\cos\theta}{\cos(30°+\theta)} - \frac{\sin\theta}{\sin(30°+\theta)}$, find θ.

30. If $\cos(\theta+\alpha) = \sin\phi \sin\beta$, and $\cos(\phi+\beta) = \sin\theta \sin\alpha$, find θ and ϕ.

Show that if ϕ_1, ϕ_2 be two values of ϕ (whose tangents are unequal), then

$$2 \cot(\phi_1 + \phi_2) = \tan\beta - \sec^2\alpha \cot\beta.$$

31. If $15(1 - \cos 2\theta \cos 2\phi) = 17 \sin 2\theta \sin 2\phi$

and $\sin(\theta-\phi) + \frac{1}{2}\sqrt{\{\sin(\theta+\phi)\sin(\theta-\phi)\}} = (\tan\theta - 1)\cos\theta\cos\phi$,

then $\tan\theta = 5$ and $\tan\phi = 3$.

32. If $\cos\theta + \cos\phi + \cos\psi = 1$

and $\sin\theta + \sin\phi + \sin\psi = 0$

and $\theta + \phi + \psi = \pi$,

find θ, ϕ, ψ.

33. If $y^2 + z^2 - 2yz\cos\alpha = d^2 \sin^2\alpha$

and $z^2 + x^2 - 2zx\cos\beta = d^2 \sin^2\beta$

and $x^2 + y^2 - 2xy\cos\gamma = d^2 \sin^2\gamma$,

find x, y, z.

CHAPTER XVI.

TRIGONOMETRICAL FACTORS.

§ 1. Algebraical Theorems.

408. Def. I. **A (rational) integral function** of any quantity is a finite sum of a finite number of terms, *involving only positive integral powers of that quantity.*

Thus $\frac{3}{2+5a}x^5+\sqrt{(5a^3)}x^2+\frac{7x-11}{9}$ is a (rational) integral function of x.

But $\frac{a}{x}+\sqrt{x}+cx^{\frac{1}{3}}+bx^{-3}$ is *not* a rational integral function of x: for the terms contain negative or fractional indices of x.

Nor is $\frac{ax^2+bx+c}{px+q}$: for the value of this fraction found by *division* would contain an *infinite* number of terms (except for special values of a, b, c, p, q).

409. Def. II. If Q and $P \div Q$ can each be expressed as a (rational) integral function of x, then Q is called an **x-factor** of P, and P is called an **x-multiple** of Q.

Thus

$x-a$ is an *x-factor* of x^2-a^2,

$\because (x^2-a^2)\div(x-a)=x+a$ (an integral function of x);

Again $ax-a^2$ is an *x-factor* of x^2-a^2,

$\because (x^2-a^2)\div(ax-a^2)=\frac{x}{a}+1$ (an integral function of x).

But $(x-a)^{-1}$ is *not* an *x-factor* of $x+a$; because although

$(x+a)\div(x-a)^{-1}=x^2-a^2$ (an integral function of x),

yet $(x-a)^{-1}$ is not *itself* an integral function of x.

410. The ordinary terms *factor* and *multiple* may be substituted for *x-factor* and *x-multiple* respectively, if it is always remembered that an *algebraic* expression can be regarded as integral or not, *only in reference to some specified symbol of quantity which it contains.*

The expression *rational integral function* is usually abbreviated into *integral function;* and the term *polynomial* is another equivalent.

411. The symbol $f(x)$, read *function* x, denotes any expression involving x; where it is understood that $f(a), f(b), f(3)$, &c. represent the values obtained by writing a, b, 3, &c. respectively, *instead of* x, in the expression denoted by $f(x)$.

Other symbols, such as $f_1(x)$, $\phi(x)$, &c., denote *other* expressions involving x, in which the substitution of a for x is denoted by $f_1(a)$, $\phi(a)$, &c.

412. Theorem I. If $f(x)$ be any integral function of x arranged in descending powers of x and divided by $x-a$, the remainder which is independent of x will be $f(a)$.

For the process of division consists in choosing such terms in the quotient as shall make each successive remainder of lower degree than the last. Hence in this way we may always obtain a remainder which does not contain x; *and which will, therefore, be unaltered by any change in the value of* x.

Let then $\phi(x)$ be the quotient and R the remainder obtained by dividing $f(x)$ by $x-a$.

Thus $$f(x) \equiv \phi(x).(x-a)+R \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

In (1) put $x=a$, then R, which does not contain x, will be unaffected.

$$\therefore f(a) \equiv \phi(a).(a-a)+R \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Now since $\phi(a)$ must be finite, $\phi(a).(a-a)=0$,

$$\therefore f(a) \equiv R \text{ or } R \equiv f(a).$$

Substituting for R in (1), we have

$$f(x) \equiv \phi(x).(x-a)+f(a) \quad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where $\phi(x)$ being the quotient is an integral function of x as well as $f(x)$.

Cor. $x-a$ is always a factor of $f(x)-f(a)$.

For, by the above,

$$f(x)-f(a) \equiv \phi(x) \,.\, (x-a)$$

where $\phi(x)$ is an integral function of x.

413. Theorem II. If $f(a_1)=f(a_2)=f(a_3)=\ldots\ldots$ and if $a_1, a_2, a_3, \ldots$ are all different, then $f(x)-f(a_1)$ is divisible by the product $(x-a_1)(x-a_2)(x-a_3)\ldots\ldots$

For, by Theorem I., dividing $f(x)-f(a_1)$ by $x-a_1$,

$$f(x)-f(a_1) \equiv \phi_1(x) \,.\, (x-a_1)+f(a_1)-f(a_1)$$

where $\phi_1(x)$ is integral.

$$\therefore f(x)-f(a_1) \equiv \phi_1(x) \,.\, (x-a_1) \quad \ldots\ldots\ldots\ldots(1).$$

Again, dividing this quotient $\phi_1(x)$ by $x-a_2$,

$$\phi_1(x) \equiv \phi_2(x) \,.\, (x-a_2)+\phi_1(a_2)$$

where $\phi_2(x)$ is integral.

But, by (1), $\phi_1(a_2) \equiv \dfrac{f(a_2)-f(a_1)}{a_2-a_1}$

$$\equiv 0, \textit{ since } a_2-a_1 \textit{ is not } 0.$$

$$\therefore \phi_1(x) \equiv \phi_2(x) \,.\, (x-a_2).$$

Substituting in (1),

$$f(x)-f(a_1) \equiv \phi_2(x) \,.\, (x-a_1)(x-a_2) \quad \ldots\ldots\ldots(2).$$

Again, dividing this quotient $\phi_2(x)$ by $x-a_3$,

$$\phi_2(x) \equiv \phi_3(x) \,.\, (x-a_3)+\phi_2(a_3)$$

where $\phi_3(x)$ is integral.

But, by (2), $\phi_2(a_3) \equiv \dfrac{f(a_3)-f(a_1)}{(a_3-a_1)(a_3-a_2)} \equiv 0,$

since a_3-a_1 *and* a_3-a_2 *are neither* $=0$.

$$\therefore \phi_2(x) \equiv \phi_3(x) \,.\, (x-a_3).$$

Substituting in (2),

$$f(x)-f(a_1) \equiv \phi_3(x) \,.\, (x-a_1)(x-a_2)(x-a_3) \quad \ldots\ldots(3).$$

In this way we may go on dividing each successive quotient by any number of factors, *which are all different*, as long as the quotient contains x at all.

Cor. If the highest term in $f(x)$ is $p \cdot x^n$, and if there are n quantities $a_1, a_2, \ldots a_n$, all different and such that

$$f(a_1) = f(a_2) = \ldots = f(a_n),$$

then $f(x) - f(a_1) \equiv p(x-a_1)(x-a_2)\ldots(x-a_n)$.

For each successive division reduces the quotient by one degree, so that the highest terms in the successive quotients are $px^{n-1}, px^{n-2}, \ldots p$.

414. Theorem III. If two integral functions of x of the n^{th} degree are equal to one another for *more than* n different values of x, then the coefficients of the different powers of x in the two functions are severally equal to one another.

For, let $\quad px^n + p'x^{n-1} + \ldots = qx^n + q'x^{n-1} + \ldots$

for the $n+1$ different values $a_1\, a_2 \ldots a_n,\ a_{n+1}$.

Then *writing* $f(x)$ *for* $(p-q)x^n + (p'-q')x^{n-1} + \ldots$

we have $\quad f(a_1) = f(a_2) = \ldots = f(a_n) = f(a_{n+1}) = 0\,;$

$$\therefore f(x) - f(a_1) = (p-q)(x-a_1)(x-a_2)\ldots(x-a_n)$$

for all values of x. Put $x = a_{n+1}$,

$$\therefore f(a_{n+1}) - f(a_1) = (p-q)(a_{n+1}-a_1)(a_{n+1}-a_2)\ldots(a_{n+1}-a_n).$$

But $f(a_{n+1}) - f(a_1) = 0$, while none of the factors $a_{n+1} - a_1$, $a_{n+1} - a_2 \ldots$ is zero.

$$\therefore\ p - q = 0, \text{ i.e. } p = q.$$

$f(x)$ is now reduced to the $(n-1)^{\text{th}}$ degree: and the same proof shows that $p' - q' = 0$, i.e. $p' = q'$ and so on.

Cor. Since $p = q$; $p' = q'$, &c.

$$\therefore\ px^n + p'x^{n-1} + \ldots = qx^n + q'x^{n-1} + \ldots$$

for all values of x. Thus, if two functions of the n^{th} degree are equal for more than n values of the variable, they are equal for *all* values of the variable.

§ 2. Application to Trigonometrical Formulæ.

415. The relation

$$2\cos(n+1)\theta + 2\cos(n-1)\theta = 2\cos\theta \,.\, 2\cos n\theta$$

shows, by induction, that $2\cos n\theta$ is an integral function of $2\cos\theta$, having $(2\cos\theta)^n$ as highest term.

For, assuming this true for n and $n-1$, we have, by above,

$$\begin{aligned} 2\cos(n+1)\theta &= 2\cos\theta \,.\, 2\cos n\theta - 2\cos(n-1)\theta \\ &= 2\cos\theta\,\{(2\cos\theta)^n + \text{lower powers}\} \\ &\qquad - \{(2\cos\theta)^{n-1} + \text{lower powers}\} \\ &= (2\cos\theta)^{n+1} + \text{lower powers.} \end{aligned}$$

Now $2\cos\theta = (2\cos\theta)^1$ and $2\cos 2\theta = (2\cos\theta)^2 - 2$.

Hence, for all positive integral values of n,

$2\cos n\theta = (2\cos\theta)^n +$ terms involving lower powers of $2\cos\theta$*.

Cor. Since, of course, $2\cos na$ is the *same* function of $2\cos a$, that $2\cos n\theta$ is of $2\cos\theta$,

$$\therefore\ 2\cos n\theta - 2\cos na \text{ } \textit{is divisible by} \text{ } 2\cos\theta - 2\cos a,$$

or $$\cos n\theta - \cos na \text{ } \textit{is divisible by} \text{ } \cos\theta - \cos a.$$

416. The relation

$$(x^{n+1} + x^{-n-1}) + (x^{n-1} + x^{-n+1}) = (x + x^{-1})(x^n + x^{-n})$$

shows precisely as in the last article, by induction, that $x^n + x^{-n}$ is an integral function of $x + x^{-1}$, having $(x + x^{-1})^n$ as highest term.

For $x + x^{-1} = (x + x^{-1})^1$ *and* $x^2 + x^{-2} = (x + x^{-1})^2 - 2$.

Cor. Since the successive expressions of $x^n + x^{-n}$ as functions of $x + x^{-1}$, and those of $2\cos n\theta$ as functions of $2\cos\theta$ are found by the same formula, and since $x^2 + x^{-2}$ is the same function of $x + x^{-1}$ as $2\cos 2\theta$ is of $2\cos\theta$, it follows that, for *all* positive integral values of n, $x^n + x^{-n}$ is the same function of $x + x^{-1}$ as $2\cos n\theta$ is of $2\cos\theta$.

Hence, for *all* positive integral values of n,

$$x^n + x^{-n} - 2\cos na \text{ } \textit{is divisible by} \text{ } x + x^{-1} - 2\cos a.$$

* This result was of course proved in Chap. XIV.

417. The relation

$$2\sin(n+1)\theta + 2\sin(n-1)\theta = 2\cos\theta \,.\, 2\sin n\theta$$

shows, by induction, that $2\sin n\theta \div 2\sin\theta$ is an integral function of $2\cos\theta$, having $(2\cos\theta)^{n-1}$ as highest term.

For, assuming this true for n and $n-1$, we have

$$\frac{2\sin(n+1)\theta}{2\sin\theta} = 2\cos\theta \,.\, \frac{2\sin n\theta}{2\sin\theta} - \frac{2\sin(n+1)\theta}{2\sin\theta}$$

$$= 2\cos\theta\,\{(2\cos\theta)^{n-1} + \text{lower powers}\}$$

$$-\{(2\cos\theta)^{n-2} + \text{lower powers}\}$$

$$= (2\cos\theta)^n + \text{lower powers.}$$

Now $$\frac{2\sin\theta}{2\sin\theta} = 1 = (2\cos\theta)^0,$$

and $$\frac{2\sin 2\theta}{2\sin\theta} = (2\cos\theta)^1.$$

Hence, for all positive integral values of n,

$$\frac{2\sin n\theta}{2\sin\theta} = (2\cos\theta)^{n-1} + \text{terms involving lower powers of } 2\cos\theta^*.$$

Cor. Hence

$$\frac{2\sin n\theta}{2\sin\theta} - \frac{2\sin n\alpha}{2\sin\alpha} \textit{ is divisible by } 2\cos\theta - 2\cos\alpha,$$

or

$$\frac{\sin n\theta}{\sin\theta} - \frac{\sin n\alpha}{\sin\alpha} \textit{ is divisible by } \cos\theta - \cos\alpha.$$

418. The relation

$$(x^{n+1} - x^{-n-1}) + (x^{n-1} - x^{-n+1}) = (x + x^{-1})(x^n - x^{-n})$$

shows, precisely as in the last article, by induction, that $\dfrac{x^n - x^{-n}}{x - x^{-1}}$ is an integral function of $x + x^{-1}$, having $(x + x^{-1})^{n-1}$ for highest term.

For $\dfrac{x - x^{-1}}{x - x^{-1}} = (x + x^{-1})^0$ *and* $\dfrac{x^2 - x^{-2}}{x - x^{-1}} = (x + x^{-1})^1$.

* This result was of course proved in Chap. XIV.

COR. Since the successive expressions of $\frac{x^n - x^{-n}}{x - x^{-1}}$ as functions of $x + x^{-1}$, and those of $\frac{2 \sin n\theta}{2 \sin \theta}$ as functions of $2 \cos \theta$ are obtained by the same formula, and since $\frac{x^2 - x^{-2}}{x - x^{-1}}$ is the same function of $x + x^{-1}$ as $\frac{2 \sin 2\theta}{2 \sin \theta}$ is of $2 \cos \theta$, it follows, that for *all* positive integral values of n, $\frac{x^n - x^{-n}}{x - x^{-1}}$ is the same function of $x + x^{-1}$ that $\frac{2 \sin n\theta}{2 \sin \theta}$ is of $2 \cos \theta$.

$$\therefore \frac{x^n - x^{-n}}{x - x^{-1}} - \frac{\sin na}{\sin a} \text{ is divisible by } x + x^{-1} - 2 \cos a.$$

419. *To completely factorise*

$$2 \cos n\theta - 2 \cos na \text{ and } x^n + x^{-n} - 2 \cos na,$$

when $\cos na = 0$.

Since $\cos na = 0$, $\therefore na = \frac{1}{2}\pi(2\lambda + 1)$

where λ has any integral value whatever.

Hence, by Arts. 415, 416, Cor.,

$$2 \cos n\theta \text{ is divisible by } 2 \cos \theta - 2 \cos \frac{\pi}{2n}(2\lambda + 1),$$

and $$x^n + x^{-n} \text{ is divisible by } x + x^{-1} - 2 \cos \frac{\pi}{2n}(2\lambda + 1).$$

Now two values of $\cos(2\lambda + 1)\pi/2n$, obtained by giving to λ different values μ and μ', cannot be equal, unless the sum or difference of the two angles is a multiple of 2π: i.e. unless

$(2\mu + 1)\pi/2n + (2\mu' + 1)\pi/2n$ or $(2\mu + 1)\pi/2n \sim (2\mu' + 1)\pi/2n$

is a multiple of 2π;

i.e. *unless* $\mu + \mu' + 1$, *or* $\mu \sim \mu'$, is a multiple of $2n$.

If then λ assumes the n values 0, 1, 2...$(n-1)$, these conditions will be excluded, so that *all* the factors so obtained will be different. Hence

$$2\cos n\theta =$$

$$\left(2\cos\theta - 2\cos\frac{\pi}{2n}\right)\left(2\cos\theta - 2\cos\frac{3\pi}{2n}\right)\ldots\left(2\cos\theta - 2\cos\frac{2n-1}{2n}\pi\right)$$

$$x^n + x^{-n} =$$

$$\left(x + x^{-1} - 2\cos\frac{\pi}{2n}\right)\left(x + x^{-1} - 2\cos\frac{3\pi}{2n}\right)\ldots\left(x + x^{-1} - 2\cos\frac{2n-1}{2n}\pi\right).$$

420. *To completely factorise*

$$\frac{\sin n\theta}{\sin\theta} - \frac{\sin na}{\sin a} \text{ and } \frac{x^n - x^{-n}}{x - x^{-1}} - \frac{\sin na}{\sin a}, \text{ when } \frac{\sin na}{\sin a} = 0.$$

Since $\sin na/\sin a = 0$, $\therefore\ na = \lambda\pi$

where λ has any integral value, *except* 0 *or a multiple of* n. These values must be excluded or $\sin a$ would vanish.

Hence, by Arts. 417, 418, Cor.,

$$\sin n\theta \textit{ is divisible by } 2\cos\theta - 2\cos\frac{\lambda\pi}{n},$$

and

$$x^n - x^{-n} \textit{ is divisible by } x + x^{-1} - 2\cos\frac{\lambda\pi}{n}.$$

Now two values of $\cos\lambda\pi/n$, obtained by giving to λ different values μ and μ', cannot be equal, unless the sum or difference of the two angles is a multiple of 2π: i.e. unless

$$\mu\pi/n + \mu'\pi/n \textit{ or } \mu\pi/n \sim \mu'\pi/n \text{ is a multiple of } 2\pi,$$

i.e. *unless* $\mu + \mu'$ *or* $\mu \sim \mu'$ is a multiple of $2n$.

If then λ assumes the $n-1$ values $1, 2 \ldots n-1$, these conditions will be excluded, so that all the factors so obtained will be different. Hence

$$\sin n\theta =$$

$$\sin\theta\left(2\cos\theta - 2\cos\frac{\pi}{n}\right)\left(2\cos\theta - 2\cos\frac{2\pi}{n}\right)\ldots\left(2\cos\theta - 2\cos\frac{n-1}{n}\pi\right),$$

$$x^n - x^{-n} =$$

$$(x - x^{-1})\left(x + x^{-1} - 2\cos\frac{\pi}{n}\right)\left(x + x^{-1} - 2\cos\frac{2\pi}{n}\right)\ldots\left(x + x^{-1} - 2\cos\frac{n-1}{n}\pi\right).$$

421. *To completely factorise* $2\cos n\theta - 2\cos na$ *and* $x^n + x^{-n} - 2\cos na$, *when* $\cos na$ *is not equal to* $+1$ *or* -1.

Since $\cos na = \cos(2\lambda\pi + na)$, where λ is integral,

$$\therefore\ 2\cos n\theta - 2\cos na \text{ is divisible by } 2\cos\theta - 2\cos\left(a + \frac{2\lambda\pi}{n}\right),$$

$$x^n + x^{-n} - 2\cos na \text{ is divisible by } x + x^{-1} - 2\cos\left(a + \frac{2\lambda\pi}{n}\right).$$

Now two values of $\cos(a + 2\lambda\pi/n)$, obtained by giving to λ different values μ and μ', cannot be equal, unless the sum or difference of the two angles is a multiple of 2π: i.e. unless

$$(a + 2\mu\pi/n) + (a + 2\mu'\pi/n) \text{ or } (a + 2\mu\pi/n) \sim (a + 2\mu'\pi/n)$$

is a multiple of 2π,

i.e. *unless* na is a multiple of π, *or* $\mu \sim \mu'$ is a multiple of n.

But if na were a multiple of π, $\cos na$ would be $+1$ or -1, which we have excluded.

Hence, if λ assumes the n values $0, 1, 2 \ldots (n-1)$, all the factors so obtained will be different.

Hence, if $\cos na$ is not $+1$ or -1,

(1) $2\cos n\theta - 2\cos na$

$$= (2\cos\theta - 2\cos a)\left\{2\cos\theta - 2\cos\left(a + \frac{2\pi}{n}\right)\right\}\ldots$$
$$\left\{2\cos\theta - 2\cos\left(a + \frac{n-1}{n}2\pi\right)\right\},$$

(2) $x^n + x^{-n} - 2\cos na$

$$= (x + x^{-1} - 2\cos a)\left\{x + x^{-1} - 2\cos\left(a + \frac{2\pi}{n}\right)\right\}\ldots$$
$$\left\{x + x^{-1} - 2\cos\left(a + \frac{n-1}{n}2\pi\right)\right\}.$$

422. *To extend the results of the last article to the excluded cases.*

I. Let $a = 0$, then $\cos na = +1$. Thus

$$2\cos n\theta - 2\cos na = 2\cos n\theta - 2 = 4\sin^2 n\,.\,\tfrac{1}{2}\theta,$$
$$x^n + x^{-n} - 2\cos na = x^n + x^{-n} - 2 = (\sqrt{x^n} - \sqrt{x^{-n}})^2.$$

Then, by Art. 420,

(1) $2\sin\frac{n\theta}{2} =$

$$2\sin\frac{\theta}{2}\left(2\cos\frac{\theta}{2}-2\cos\frac{\pi}{n}\right)\left(2\cos\frac{\theta}{2}-2\cos\frac{2\pi}{n}\right)\ldots\left(2\cos\frac{\theta}{2}-2\cos\frac{n-1}{n}\pi\right).$$

(2) $\sqrt{x^n}-\sqrt{x^{-n}} =$

$$(\sqrt{x}-\sqrt{x^{-1}})\left(\sqrt{x}+\sqrt{x^{-1}}-2\cos\frac{\pi}{n}\right)\ldots\left(\sqrt{x}+\sqrt{x^{-1}}-2\cos\frac{n-1}{n}\pi\right).$$

II. Let $a=\pi/n$, then $\cos na=-1$. Thus

$$2\cos n\theta-2\cos na=2\cos n\theta+2=4\cos^2 n\,.\,\tfrac{1}{2}\theta,$$
$$x^n+x^{-n}-2\cos na=x^n+x^{-n}+2=(\sqrt{x^n}+\sqrt{x^{-n}})^2.$$

Then, by Art. 419,

(3) $2\cos\frac{n\theta}{2} =$

$$\left(2\cos\frac{\theta}{2}-2\cos\frac{\pi}{2n}\right)\left(2\cos\frac{\theta}{2}-2\cos\frac{3\pi}{2n}\right)\ldots\left(2\cos\frac{\theta}{2}-2\cos\frac{2n-1}{2n}\pi\right).$$

(4) $\sqrt{x^n}+\sqrt{x^{-n}} =$

$$\left(\sqrt{x}+\sqrt{x^{-1}}-2\cos\frac{\pi}{2n}\right)\ldots\left(\sqrt{x}+\sqrt{x^{-1}}-2\cos\frac{2n-1}{2n}\pi\right).$$

Now, since $\cos\lambda\pi/2n=-\cos(2n-\lambda)\pi/2n$,

$$\therefore\left(2\cos\frac{\theta}{2}-2\cos\frac{\lambda\pi}{2n}\right)\left(2\cos\frac{\theta}{2}-2\cos\frac{2n-\lambda}{2n}\pi\right)$$
$$=4\cos^2\frac{\theta}{2}-4\cos^2\frac{\lambda\pi}{2n}=2\cos\theta-2\cos\frac{\lambda\pi}{n},$$

$$\left(\sqrt{x}+\sqrt{x^{-1}}-2\cos\frac{\lambda\pi}{2n}\right)\left(\sqrt{x}+\sqrt{x^{-1}}-2\cos\frac{2n-\lambda}{2n}\pi\right)$$
$$=x+2+x^{-1}-4\cos^2\frac{\lambda\pi}{2n}=x+x^{-1}-2\cos\frac{\lambda\pi}{n}.$$

Hence, by multiplying each of the above four series of factors by the same series *inverted*, we see that the results of the last article are applicable to the excluded cases in which $\cos na=\pm 1$.

423. From Arts. 419, 420, we have

$$x^m + x^{-m} =$$

$$\left(x+x^{-1}-2\cos\frac{\pi}{2m}\right)\left(x+x^{-1}-2\cos\frac{3\pi}{2m}\right)\ldots\left(x+x^{-1}-2\cos\frac{2m-1}{2m}\pi\right),$$

$$x^m - x^{-m} = (x-x^{-1})\times$$

$$\left(x+x^{-1}-2\cos\frac{\pi}{m}\right)\left(x+x^{-1}-2\cos\frac{2\pi}{m}\right)\ldots\left(x+x^{-1}-2\cos\frac{m-1}{m}\pi\right).$$

Multiply both sides of these equations by x^m; and let $n=2m$.

Thus, *if n is even*,

(1) $x^n + 1 =$

$$\left(x^2-2x\cos\frac{\pi}{n}+1\right)\left(x^2-2x\cos\frac{3\pi}{n}+1\right)\ldots\left(x^2-2x\cos\frac{n-1}{n}\pi+1\right).$$

(2) $x^n - 1 = (x^2-1)\times$

$$\left(x^2-2x\cos\frac{2\pi}{n}+1\right)\left(x^2-2x\cos\frac{4\pi}{n}+1\right)\ldots\left(x^2-2x\cos\frac{n-2}{n}\pi+1\right).$$

Now $(x^2+1-2x\cos\lambda\pi/n)\{x^2+1-2x\cos(n-\lambda)\pi/n\}$

$$= (x^2+1)^2 - 4x^2\cos^2\lambda\pi/n = x^4 - 2x^2\cos 2\lambda\pi/n + 1.$$

Hence we may combine the first and last, second and last but one, &c. of the factors in (1) and (2) which involve a cosine.

If then we suppose $\frac{1}{2}n$ or m to be *odd*, there will be an unpaired factor in (1): viz. $x^2 - 2x\cos\frac{1}{2}\pi + 1 = x^2 + 1$.

Putting $x^2 = y$, we have

If m is odd,

(3) $y^m + 1 = (y+1)\times$

$$\left(y^2-2y\cos\frac{\pi}{m}+1\right)\left(y^2-2y\cos\frac{3\pi}{m}+1\right)\ldots\left(y^2-2y\cos\frac{m-2}{m}\pi+1\right).$$

(4) $y^m - 1 = (y-1)\times$

$$\left(y^2-2y\cos\frac{2\pi}{m}+1\right)\left(y^2-2y\cos\frac{4\pi}{m}+1\right)\ldots\left(y^2-2y\cos\frac{m-1}{m}\pi+1\right).$$

The formulæ (1), (2), (3), (4) are highly important.

424. Again, from Arts. 419, 420, we have

(1) $\cos n\theta =$

$$2^{n-1}\left(\cos\theta-\cos\frac{\pi}{2n}\right)\left(\cos\theta-\cos\frac{3\pi}{2n}\right)\ldots\left(\cos\theta-\cos\frac{2n-1}{2n}\pi\right).$$

(2) $\sin n\theta =$

$$2^{n-1}\sin\theta\left(\cos\theta-\cos\frac{\pi}{n}\right)\left(\cos\theta-\cos\frac{2\pi}{n}\right)\ldots\left(\cos\theta-\cos\frac{n-1}{n}\pi\right).$$

These factor-formulæ may be put into other forms.

For $$\begin{aligned}&(\cos\theta-\cos\lambda\pi/2n)\{\cos\theta-\cos(2n-\lambda)\pi/2n\}\\&=(\cos\theta-\cos\lambda\pi/2n)(\cos\theta+\cos\lambda\pi/2n)\\&=\cos^2\theta-\cos^2\lambda\pi/2n=\sin^2\lambda\pi/2n-\sin^2\theta.\end{aligned}$$

If then, in (1), *n is even:* and if, in (2), *n is odd*, each of the binomial factors can be paired. Thus

(3) n even, $\cos n\theta =$

$$2^{n-1}\left(\sin^2\frac{\pi}{2n}-\sin^2\theta\right)\left(\sin^2\frac{3\pi}{2n}-\sin^2\theta\right)\ldots\left(\sin^2\frac{n-1}{2n}\pi-\sin^2\theta\right).$$

(4) n odd, $\sin n\theta =$

$$2^{n-1}\sin\theta\left(\sin^2\frac{\pi}{n}-\sin^2\theta\right)\left(\sin^2\frac{2\pi}{n}-\sin^2\theta\right)\ldots\left(\sin^2\frac{n-1}{2n}\pi-\sin^2\theta\right).$$

But if, in (1), *n is odd:* and if, in (2), *n is even*, each of the binomial factors, *except the middle*, can be paired. Thus

(5) n odd, $\cos n\theta =$

$$2^{n-1}\cos\theta\left(\sin^2\frac{\pi}{2n}-\sin^2\theta\right)\left(\sin^2\frac{3\pi}{2n}-\sin^2\theta\right)\ldots\left(\sin^2\frac{n-2}{2n}\pi-\sin^2\theta\right).$$

(6) n even, $\sin n\theta =$

$$2^{n-1}\sin\theta\cos\theta\left(\sin^2\frac{\pi}{n}-\sin^2\theta\right)\left(\sin^2\frac{2\pi}{n}-\sin^2\theta\right)\ldots\left(\sin^2\frac{n-2}{2n}\pi-\sin^2\theta\right).$$

Lastly, since $$\begin{aligned}\sin^2\lambda\pi/2n-\sin^2\theta&=\sin(\lambda\pi/2n+\theta)\sin(\lambda\pi/2n-\theta)\\&=\sin(\lambda\pi/2n+\theta)\sin\{(2n-\lambda)\pi/2n+\theta\}.\end{aligned}$$

And since in (5) and (6), $\cos\theta = \sin(\frac{1}{2}\pi + \theta)$,

we have, for *all* values of n,

$$(7)\quad \cos n\theta = 2^{n-1}\sin\left(\theta + \frac{\pi}{2n}\right)\sin\left(\theta + \frac{3\pi}{2n}\right)\ldots\sin\left(\theta + \frac{2n-1}{2n}\pi\right).$$

$$(8)\quad \sin n\theta = 2^{n-1}\sin\theta\sin\left(\theta + \frac{\pi}{n}\right)\sin\left(\theta + \frac{2\pi}{n}\right)\ldots\sin\left(\theta + \frac{n-1}{n}\pi\right).$$

The formulæ (1), (2), (7), (8) can easily be remembered, by giving to θ the several values which will make both sides of the identity vanish.

§ 3. Deductions from the Factor-Formulæ.

425. By comparing the expansions for $2\cos n\theta$ and for $\sin n\theta/\sin\theta$ in the last chapter, with the expansions obtained by the factor-formulæ of the present chapter, we may obtain many important results.

Thus, equating coefficients (by Art. 414) of the powers of $2\cos\theta$ in the two expansions of $\sin n\theta \div \sin\theta$,

the $(n-1)$ quantities $2\cos\pi/n$, $2\cos 2\pi/n$, $\ldots 2\cos(n-1)\pi/n$,

are such that their sum $= 0$;

the sum of their products, *two* together $= -(n-2)$;

the sum of their products, *three* together $= 0$;

the sum of their products, *four* together $= \frac{1}{2}(n-3)(n-4)$;

and so on.

Equating the coefficients of the powers of $2\cos\theta$ in the two expansions of $2\cos n\theta$,

the n quantities $2\cos\pi/2n$, $2\cos 3\pi/2n, \ldots 2\cos(2n-1)\pi/2n$

are such that their sum $= 0$;

the sum of their products, *two* together $= -n$;

the sum of their products, *three* together $= 0$;

the sum of their products, *four* together $= \frac{1}{2}n(n-3)$;

and so on.

426. In Arts. 419, 420 put $x = 1$.

Then since

$$\frac{x^n - x^{-n}}{x - x^{-1}} = x^{n-1} + x^{n-3} + \ldots + x^{-n+1}.$$

$\therefore$ writing ϕ for *one nth of a right-angle.*

$2 = (2 - 2\cos\phi)(2 - 2\cos 3\phi)\ldots$*up to* $\{2 - 2\cos(2n-1)\phi\}\ldots(1)$,

$n = (2 - 2\cos 2\phi)(2 - 2\cos 4\phi)\ldots$*up to* $\{2 - 2\cos(2n-2)\phi\}\ldots(2)$.

Take the square root. Thus

$$\sqrt{2} = 2^n \sin\tfrac{1}{2}\phi \,.\, \sin\tfrac{1}{2}3\phi\ldots \textit{up to } \sin\tfrac{1}{2}(2n-1)\phi\ldots\ldots(3),$$

$$\sqrt{n} = 2^{n-1} \sin\tfrac{1}{2}2\phi \,.\, \sin\tfrac{1}{2}4\phi\ldots \textit{up to } \sin\tfrac{1}{2}(2n-2)\phi\ldots(4).$$

Now $\sin\frac{1}{2}\lambda\phi = \cos\frac{1}{2}(2n-\lambda)\phi$. Thus

$$\sqrt{2} = 2^n \cos\tfrac{1}{2}\phi \,.\, \cos\tfrac{1}{2}3\phi\ldots \textit{up to } \cos\tfrac{1}{2}(2n-1)\phi\ldots\ldots(5),$$

$$\sqrt{n} = 2^{n-1} \cos\tfrac{1}{2}2\phi \,.\, \cos\tfrac{1}{2}4\phi\ldots \textit{up to } \cos\tfrac{1}{2}(2n-2)\phi\ldots(6).$$

Hence, multiplying (3) by (5); and (4) by (6);

$$1 = 2^{n-1} \sin\phi \sin 3\phi\ldots \textit{up to } \sin(2n-1)\phi\ldots\ldots\ldots(7),$$

$$n = 2^{n-1} \sin 2\phi \sin 4\phi\ldots \textit{up to } \sin(2n-2)\phi\ldots\ldots\ldots(8).$$

Again, $\sin\lambda\phi = \sin(2n-\lambda)\phi$.

Hence, taking the square root,

$$1 = \sqrt{2^{n-1}} \,.\, \sin\phi \sin 3\phi \sin 5\phi\ldots\&c.\ldots\ldots\ldots\ldots(9),$$

$$\sqrt{n} = \sqrt{2^{n-1}} \,.\, \sin 2\phi \sin 4\phi \sin 6\phi\ldots\&c.\ldots\ldots\ldots\ldots(10),$$

where the coefficients of ϕ have every *odd* or *even* value *less than* n. For the unpaired factor $\sin n\phi$ (when it occurs) $= 1$.

427. *To apply geometrically the factorisation of*

$$2\cos n\theta - 2\cos n\alpha \textit{ and } x^n + x^{-n} - 2\cos n\alpha.$$

Take any circle with centre O. Draw the radius OI.

Let A be any point on the circumference, and angle $IOA = \alpha$.

Divide the whole circumference into n equal arcs, AB, BC, CD ... beginning at A: so that, if L is any one of the points of division,

$$\text{angle } IOL = \alpha + 2\lambda\pi/n.$$

Also $$OA = OB = \ldots = OL = \ldots = OI.$$

I. Let Q be any point on the circumference, such that angle $IOQ = \theta$: and let Qq, Aa, Bb,... be perpendiculars upon OI. Then will

$$OI^n(\cos n\theta - \cos n\alpha) = 2^{n-1} \cdot aq \cdot bq \cdot cq \ldots \text{ to } n \text{ factors} \quad ..(1).$$

For $OQ \cos\theta - OL \cos(\alpha + 2\lambda\pi/n) = Oq - Ol = lq$.

$\therefore$, multiplying $\cos n\theta - \cos n\alpha$ by OI^n, we have (1).

II. Let P be any point on OI, such that $OP = x \cdot OI$; and join PA, PB, PC .... Then will

$$OP^{2n} - 2OP^n \cdot OI^n \cos n\alpha + OI^{2n} = PA^2 \cdot PB^2 \cdot PC^2 \ldots \text{to } n \text{ factors} \ldots (2).$$

For $PL^2 = OP^2 + OL^2 - 2OP \cdot OL \cos IOL$

$$= OI^2\{x^2 + 1 - 2x\cos(\alpha + 2\lambda\pi/n)\}.$$

$\therefore$, multiplying $x^n + x^{-n} - 2\cos n\alpha$ by $x^n \cdot OI^{2n}$, we have (2).

428. Particular cases of the two theorems, proved above, should be noted. Thus

(1) Let Q and P coincide with I.

Then $\theta = 0$ and $x = 1$. Thus

$$OI^n \cdot \sin^2 \tfrac{1}{2} n\alpha = 2^{n-2} \cdot aI \cdot bI \cdot cI \ldots \text{ to } n \text{ factors.}$$
$$2OI^n \cdot \sin \tfrac{1}{2} n\alpha = AI \cdot BI \cdot CI \ldots\ldots \text{ to } n \text{ factors.}$$

(2) Let A coincide with I.

Then $\alpha = 0$. Thus

$$-OI^n \cdot \sin^2 \tfrac{1}{2} n\theta = 2^{n-2} \cdot aq \cdot bq \cdot cq \ldots \text{ to } n \text{ factors.}$$
$$OP^n \sim OI^n = AP \cdot BP \cdot CP \ldots \text{ to } n \text{ factors.}$$

(3) Let the arcs AB, BC, CD ... in (2) be bisected in A', B', C' .... Then

$$-OI^{2n} \cdot \sin^2 n\theta = 2^{2n-2} \cdot aq \cdot a'q \cdot bq \cdot b'q \ldots \text{ to } 2n \text{ factors.}$$
$$OP^{2n} \sim OI^{2n} = AP \cdot A'P \cdot BP \cdot B'P \ldots \text{ to } 2n \text{ factors.}$$

$\therefore$ by division,

$$OI^n \cos^2 \tfrac{1}{2} n\theta = 2^{n-2} \cdot a'q \cdot b'q \cdot c'q \ldots \text{ to } n \text{ factors.}$$
$$OP^n + OI^n = A'P \cdot B'P \cdot C'P \ldots \text{ to } n \text{ factors.}$$

429. The result (2) of Art. 427 is called *De Moivre's property of the circle*: and the particular cases of it in Art. 428 are called *Cotes's properties of the circle.*

430. We have shown that $x^n + x^{-n}$ is the same function of $x + x^{-1}$ as $2\cos n\theta$ is of $2\cos\theta$.

It follows that $(2\cos\theta)^n$ can be expanded in cosines of multiples of θ according to the same formula by which $(x + x^{-1})^n$ can be expanded in functions of the form $x^n + x^{-n}$. Thus, by the binomial theorem,

$$(x + x^{-1})^n = x^n + nx^{n-1} \cdot x^{-1} + \tfrac{1}{2}n(n-1)x^{n-2} \cdot x^2 + \ldots + \tfrac{1}{2}n(n-1)x^2 \cdot x^{-n+2} + nx \cdot x^{-n+1} + x^{-n}.$$

Writing the series again in the reverse order; and *adding:* we have

$$2(x + x^{-1})^n = (x^n + x^{-n}) + n(x^{n-2} + x^{-n+2}) + \tfrac{1}{2}n(n-1)(x^{n-4} + x^{-n+4}) + \ldots$$

Hence

$$(2\cos\theta)^n = \cos n\theta + n\cos(n-2)\theta + \ldots + n\cos(-n+2)\theta + \cos(-n\theta).$$

This is another mode of proving the result of Art. 362.

EXAMPLES XVI.

1. Show that, if $f(x)$ is an integral function of degree not less than n, then

$$f(x) = \phi(x)(x-a_1)(x-a_2)\ldots(x-a_n) + f(a_1)\frac{(x-a_2)(x-a_3)\ldots(x-a_n)}{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)} + f(a_2)\frac{(x-a_1)(x-a_3)\ldots(x-a_n)}{(a_2-a_1)(a_2-a_3)\ldots(a_2-a_n)} + \&c.$$

where $\phi(x)$ is some integral function of x, and $a_1, a_2, a_3 \ldots a_n$ are any n quantities, all different.

2. Show that the sum of the coefficients of $f(a_1), f(a_2) \ldots f(a_n)$ in the above question is unity.

3. If $a =$ one *nth* of half a right angle,

$$\sin a \sin 5a \sin 9a \ldots \sin(4n-3)a = \sqrt{2} \cdot 2^{-n}.$$

4. The product of all the lines that can be drawn from one of the angles of a regular polygon inscribed in a circle whose radius is a to all the other angular points is na^{n-1}.

5. If $p_1, p_2 \ldots p_{2n-1}, p_{2n}$ be the perpendiculars drawn from any point in the circumference of a circle of radius a on the sides of a regular circumscribing polygon of $2n$ sides, then

$$p_1 p_3 p_5 \ldots p_{2n-1} + p_2 p_4 \ldots p_{2n} = a^n \cdot 2^{-n+2}.$$

6. A polygon is described about a circle touching it at the angular points of an inscribed polygon: the product of the perpendiculars drawn to the several sides of the inscribed polygon from any point in the circumference of the circle is equal to the product of the perpendiculars drawn from the same point to the several sides of the circumscribed polygon.

7. AB is the diameter of a circle of centre O, and Q_0 any point on the circumference; Q_1, Q_2, Q_3...are the points of bisection of the arcs AQ_0, AQ_1, AQ_2..., prove that

$$BQ_1 \cdot BQ_2 \cdot BQ_3 \ldots BQ_n = OA^n \cdot AQ_0 / AQ_n.$$

8. Prove that

$$\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14} = \frac{1}{8}, \text{ and } \sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \frac{\sqrt{7}}{8}.$$

9. Find the equations whose roots shall be

(*a*) $2\cos\frac{2}{7}\pi$, $2\cos\frac{4}{7}\pi$, $2\cos\frac{6}{7}\pi$. (*b*) $\sin^2\frac{1}{7}\pi$, $\sin^2\frac{3}{7}\pi$, $\sin^2\frac{5}{7}\pi$.

10. Show that $x^{12} + x^{10} + x^8 + x^6 + x^4 + x^2 + 1$

$$= \left(x^2 - 2x\cos\frac{\pi}{7} + 1\right)\left(x^2 - 2x\cos\frac{2\pi}{7} + 1\right)\ldots\left(x^2 - 2x\cos\frac{6\pi}{7} + 1\right).$$

11. If $\theta = \dfrac{\pi}{2^n + 1}$, $2^n \cos\theta \cos 2\theta \cos 2^2\theta \ldots \cos 2^{n-1}\theta = 1$.

12. If n is even, and if $n\phi = \pi$,

$$\tan a \tan(a + \phi)\tan(a + 2\phi)\ldots\tan\{a + (n-1)\phi\} = (-1)^{\frac{n}{2}}.$$

13. Prove that

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{2^7}.$$

14. From Ex. 26, p. 278, show that

$$\cot\theta + \cot\left(\theta + \frac{\pi}{n}\right) + \ldots + \cot\left(\theta + \frac{n-1}{n}\pi\right) = n\cot n\theta.$$

CHAPTER XVII.

SUMMATION OF SERIES.

The difference method.

431. *To find the sum of the sines or cosines of angles* in Arithmetical Progression.

1. Let $S \equiv \sin a + \sin(a+\delta) + \ldots + \sin\{a+(n-1)\delta\}$.

Multiply each term by $2\sin$ ($\frac{1}{2}$ the common difference).

Thus $2\sin a \sin\frac{1}{2}\delta = \cos(a-\frac{1}{2}\delta) - \cos(a+\frac{1}{2}\delta)$,

$2\sin(a+\delta)\sin\frac{1}{2}\delta = \cos(a+\frac{1}{2}\delta) - \cos(a+\frac{1}{2}3\delta)$

and so on, up to

$2\sin\{a+(n-1)\delta\}\sin\frac{1}{2}\delta = \cos\{a+\frac{1}{2}(2n-3)\delta\} - \cos\{a+\frac{1}{2}(2n-1)\delta\}$

$\therefore$ by addition

$$2S.\sin\tfrac{1}{2}\delta = \cos(a-\tfrac{1}{2}\delta) - \cos\{a+\tfrac{1}{2}(2n-1)\delta\}$$
$$= 2\sin\{a+\tfrac{1}{2}(n-1)\delta\}.\sin\tfrac{1}{2}n\delta,$$
$$\therefore S = \frac{\sin\{a+\frac{1}{2}(n-1)\delta\}.\sin\frac{1}{2}n\delta}{\sin\frac{1}{2}\delta}.$$

2. Let $S \equiv \cos a + \cos(a+\delta) + \ldots + \cos\{a+(n-1)\delta\}$.

As before, multiply each term by $2\sin$ ($\frac{1}{2}$ the common difference). Thus we shall find

$$S = \frac{\cos\{a+\frac{1}{2}(n-1)\delta\}.\sin\frac{1}{2}n\delta}{\sin\frac{1}{2}\delta}.$$

432. *To find the sum of the sines or cosines,* alternately negatived, *of angles in arithmetical progression.*

1. Let

$$S \equiv \sin a - \sin(a+\delta) + \sin(a+2\delta) \ldots + (-1)^{n-1}\sin\{a+(n-1)\delta\}.$$

Multiply each term by $2\cos(\frac{1}{2}$ the common difference).

Thus $2\sin a\cos\frac{1}{2}\delta = \sin(a+\frac{1}{2}\delta) + \sin(a-\frac{1}{2}\delta)$

$$2\sin(a+\delta)\cos\tfrac{1}{2}\delta = \sin(a+\tfrac{1}{2}3\delta) + \sin(a+\tfrac{1}{2}\delta)$$

and so on, up to

$$2\sin\{a+(n-1)\delta\}\cos\tfrac{1}{2}\delta = \sin\{a+\tfrac{1}{2}(2n-1)\delta\} + \sin\{a+\tfrac{1}{2}(2n-3)\delta\}$$

$$\therefore\ 2S\,.\cos\tfrac{1}{2}\delta = \sin(a-\tfrac{1}{2}\delta) + (-1)^{n-1}\sin\{a+\tfrac{1}{2}(2n-1)\delta\},$$

$$\therefore\ S\,.\cos\tfrac{1}{2}\delta = \sin\{a+\tfrac{1}{2}(n-1)\delta\}\cos\tfrac{1}{2}n\delta, \textit{ if } n \textit{ is odd},$$

$$= -\cos\{a+\tfrac{1}{2}(n-1)\delta\}\sin\tfrac{1}{2}n\delta, \textit{ if } n \textit{ is even}.$$

2. Let

$$S \equiv \cos a - \cos(a+\delta) + \cos(a+2\delta) \ldots + (-1)^{n-1}\cos\{a+(n-1)\delta\}.$$

Then, as in (1),

$$S\,.\cos\tfrac{1}{2}\delta = \cos\{a+\tfrac{1}{2}(n-1)\delta\}\cos\tfrac{1}{2}n\delta, \textit{ if } n \textit{ is odd},$$

$$= \sin\{a+\tfrac{1}{2}(n-1)\delta\}\sin\tfrac{1}{2}n\delta, \textit{ if } n \textit{ is even}.$$

433. To find

$$S \equiv \operatorname{cosec} 2a + \operatorname{cosec} 2^2a + \ldots + \operatorname{cosec} 2^n a.$$

We have (see also p. 79)

$$\operatorname{cosec} 2a + \cot 2a = \frac{1+\cos 2a}{\sin 2a} = \frac{2\cos^2 a}{2\sin a\cos a} = \cot a.$$

$$\therefore\ \operatorname{cosec} 2a = \cot a - \cot 2a,$$

$$\therefore\ \operatorname{cosec} 2^2a = \cot 2a - \cot 2^2a,$$

and so on, up to

$$\operatorname{cosec} 2^n a = \cot 2^{n-1}a - \cot 2^n a,$$

$\therefore$ by addition, $$S = \cot a - \cot 2^n a.$$

434. To find

$$S \equiv \tan a + 2\tan 2a + 2^2\tan 2^2a + \ldots + 2^{n-1}\tan 2^{n-1}a.$$

We have (see also p. 79)

$$\cot a - \tan a = \frac{\cos^2 a - \sin^2 a}{\sin a\cos a} = \frac{2\cos 2a}{\sin 2a} = 2\cot 2a.$$

$$\therefore\ \tan a = \cot a - 2\cot 2a,$$

$$2\tan 2a = 2\cot 2a - 2^2\cot 2^2a$$

and so on, up to

$$2^{n-1}\tan 2^{n-1}a = 2^{n-1}\cot 2^{n-1}a - 2^n \cot 2^n a,$$

$\therefore$ by addition, $S = \cot a - 2^n \cot 2^n a$.

435. To find

$$S \equiv \sin(a+\beta) + 2\sin(a+2\beta) + \ldots + r\sin(a+r\beta) + \ldots + n\sin(a+n\beta).$$

Multiply each term by $2 - 2\cos\beta$. Thus

$$(2-2\cos\beta)\, r\sin(a+r\beta) = -r\sin\{a+(r-1)\beta\} + 2r\sin(a+r\beta) - r\sin\{a+(r+1)\beta\}.$$

Thus, adding all the terms, we have

$$2S(1-\cos\beta) = -\sin a + (n+1)\sin(a+n\beta) - n\sin(a+n\beta+\beta).$$

436. Similarly, if

$$S \equiv \cos(a+\beta) + 2\cos(a+2\beta) + \ldots + r\cos(a+r\beta) + \ldots + n\cos(a+n\beta)$$

$$2S(1-\cos\beta) = -\cos a + (n+1)\cos(a+n\beta) - n\cos(a+n\beta+\beta).$$

Recurring Method.

437. *To find the sum of the following series:*

Let $S = \sin a + x\sin(a+\beta) + \ldots + x^{r+1}\sin\{a+(r+1)\beta\} + \ldots + x^{n-1}\sin\{a+(n-1)\beta\}$.

Then $S \cdot 2x\cos\beta = 2x\sin a\cos\beta + \ldots + 2x^{r+1}\sin(a+r\beta)\cos\beta + \ldots + 2x^n\sin\{a+(n-1)\beta\}\cos\beta$.

And $S \cdot x^2 = \ldots + x^{r+1}\sin\{a+(r-1)\beta\} + \ldots + x^n\sin\{a+(n-2)\beta\} + x^{n+1}\sin\{a+(n-1)\beta\}$.

$\therefore S(1-2x\cos\beta+x^2) =$

$$\sin a - x\sin(a-\beta) - x^n\sin(a+n\beta) + x^{n+1}\sin\{a+(n-1)\beta\}.$$

438. *To find the sum of the following series:*

Let $S = \cos a + x\cos(a+\beta) + \ldots + x^{r+1}\cos\{a+(r+1)\beta\} + \ldots + x^{n-1}\cos\{a+(n-1)\beta\}$.

Then $S \cdot 2x\cos\beta = 2x\cos a\cos\beta + \ldots + 2x^{r+1}\cos(a+r\beta)\cos\beta + \ldots + 2x^n\cos\{a+(n-1)\beta\}\cos\beta$.

And $S \cdot x^2 = \ldots + x^{r+1}\cos\{\alpha + (r-1)\beta\} + \ldots$
$+ x^n \cos\{\alpha + (n-2)\beta\} + x^{n+1}\cos\{\alpha + (n-1)\beta\}.$
$\therefore\ S(1 - 2x\cos\beta + x^2) =$
$\cos\alpha - x\cos(\alpha - \beta) - x^n\cos(\alpha + n\beta) + x^{n+1}\cos\{\alpha + (n-1)\beta\}.$

EXAMPLES XVII.

1. If $\phi = 2\pi/n$,
$\sin\alpha + \sin(\alpha + \phi) + \ldots + \sin\{\alpha + (n-1)\phi\} = 0,$
$\cos\alpha + \cos(\alpha + \phi) + \ldots + \cos\{\alpha + (n-1)\phi\} = 0.$

2. If $\phi = \pi/n$, and n is odd,
$\sin\alpha - \sin(\alpha + \phi) + \ldots + \sin\{\alpha + (n-1)\phi\} = 0,$
$\cos\alpha - \cos(\alpha + \phi) + \ldots + \cos\{\alpha + (n-1)\phi\} = 0.$

3. $\tan\frac{1}{2}(n+1)\alpha = \dfrac{\sin\alpha + \sin 2\alpha + \sin 3\alpha + \ldots + \sin n\alpha}{\cos\alpha + \cos 2\alpha + \cos 3\alpha + \ldots + \cos n\alpha}.$

4. $\tan n\alpha = \dfrac{\sin\alpha + \sin 3\alpha + \sin 5\alpha + \ldots + \sin(2n-1)\alpha}{\cos\alpha + \cos 3\alpha + \cos 5\alpha + \ldots + \cos(2n-1)\alpha}.$

5. $\tan\frac{1}{2}(n+1)(\pi + \alpha) = \dfrac{\sin\alpha - \sin 2\alpha + \sin 3\alpha - \ldots \text{ to } n \text{ terms}}{\cos\alpha - \cos 2\alpha + \cos 3\alpha - \ldots \text{ to } n \text{ terms}}.$

Sum to n terms the following series:

6. $\cos^2\alpha + \cos^2(\alpha + \beta) + \cos^2(\alpha + 2\beta) + \ldots.$

7. $\sin^3\alpha + \sin^3(\alpha + \beta) + \sin^3(\alpha + 2\beta) + \ldots.$

8. $\cos^4\alpha + \cos^4(\alpha + \beta) + \cos^4(\alpha + 2\beta) + \ldots.$

9. $\cos\theta\cos(\theta + \alpha) + \cos(\theta + \alpha)\cos(\theta + 2\alpha)$
$+ \cos(\theta + 2\alpha)\cos(\theta + 3\alpha) + \ldots.$

10. $\sin 2\theta\cos\theta + \sin 3\theta\cos 2\theta + \sin 4\theta\cos 3\theta + \ldots.$

11. $\sec\alpha\sec 2\alpha + \sec 2\alpha\sec 3\alpha + \sec 3\alpha\sec 4\alpha + \ldots$

12. $\dfrac{1}{\cos\alpha + \cos 3\alpha} + \dfrac{1}{\cos\alpha + \cos 5\alpha} + \dfrac{1}{\cos\alpha + \cos 7\alpha} + \ldots.$

13. $\sin^2\alpha\sin 2\alpha + \frac{1}{2}\sin^2 2\alpha\sin 4\alpha + \frac{1}{4}\sin^2 4\alpha\sin 8\alpha + \ldots.$

and so on, up to

$$2^{n-1}\tan 2^{n-1}a = 2^{n-1}\cot 2^{n-1}a - 2^n \cot 2^n a,$$

$\therefore$ by addition, $S = \cot a - 2^n \cot 2^n a.$

435. To find

$$S \equiv \sin(a+\beta) + 2\sin(a+2\beta) + \ldots + r\sin(a+r\beta) + \ldots + n\sin(a+n\beta).$$

Multiply each term by $2 - 2\cos\beta$. Thus

$$(2-2\cos\beta)\, r\sin(a+r\beta) = -r\sin\{a+(r-1)\beta\} + 2r\sin(a+r\beta) - r\sin\{a+(r+1)\beta\}.$$

Thus, adding all the terms, we have

$$2S(1-\cos\beta) = -\sin a + (n+1)\sin(a+n\beta) - n\sin(a+n\beta+\beta).$$

436. Similarly, if

$$S \equiv \cos(a+\beta) + 2\cos(a+2\beta) + \ldots + r\cos(a+r\beta) + \ldots + n\cos(a+n\beta)$$

$$2S(1-\cos\beta) = -\cos a + (n+1)\cos(a+n\beta) - n\cos(a+n\beta+\beta).$$

Recurring Method.

437. *To find the sum of the following series:*

Let $S = \sin a + x\sin(a+\beta) + \ldots + x^{r+1}\sin\{a+(r+1)\beta\} + \ldots + x^{n-1}\sin\{a+(n-1)\beta\}.$

Then $S \,.\, 2x\cos\beta = 2x\sin a\cos\beta + \ldots + 2x^{r+1}\sin(a+r\beta)\cos\beta + \ldots + 2x^n\sin\{a+(n-1)\beta\}\cos\beta.$

And $S \,.\, x^2 = \ldots + x^{r+1}\sin\{a+(r-1)\beta\} + \ldots + x^n\sin\{a+(n-2)\beta\} + x^{n+1}\sin\{a+(n-1)\beta\}.$

$$\therefore S(1-2x\cos\beta+x^2) =$$

$$\sin a - x\sin(a-\beta) - x^n\sin(a+n\beta) + x^{n+1}\sin\{a+(n-1)\beta\}.$$

438. *To find the sum of the following series:*

Let $S = \cos a + x\cos(a+\beta) + \ldots + x^{r+1}\cos\{a+(r+1)\beta\} + \ldots + x^{n-1}\cos\{a+(n-1)\beta\}.$

Then $S \,.\, 2x\cos\beta = 2x\cos a\cos\beta + \ldots + 2x^{r+1}\cos(a+r\beta)\cos\beta + \ldots + 2x^n\cos\{a+(n-1)\beta\}\cos\beta.$

And $S \cdot x^2 = \ldots + x^{r+1}\cos\{\alpha + (r-1)\beta\} + \ldots$
$+ x^n \cos\{\alpha + (n-2)\beta\} + x^{n+1}\cos\{\alpha + (n-1)\beta\}.$
$\therefore S(1 - 2x\cos\beta + x^2) =$
$\cos\alpha - x\cos(\alpha - \beta) - x^n\cos(\alpha + n\beta) + x^{n+1}\cos\{\alpha + (n-1)\beta\}.$

EXAMPLES XVII.

1. If $\phi = 2\pi/n$,
$\sin\alpha + \sin(\alpha + \phi) + \ldots + \sin\{\alpha + (n-1)\phi\} = 0,$
$\cos\alpha + \cos(\alpha + \phi) + \ldots + \cos\{\alpha + (n-1)\phi\} = 0.$

2. If $\phi = \pi/n$, and n is odd,
$\sin\alpha - \sin(\alpha + \phi) + \ldots + \sin\{\alpha + (n-1)\phi\} = 0,$
$\cos\alpha - \cos(\alpha + \phi) + \ldots + \cos\{\alpha + (n-1)\phi\} = 0.$

3. $\tan\frac{1}{2}(n+1)\alpha = \dfrac{\sin\alpha + \sin 2\alpha + \sin 3\alpha + \ldots + \sin n\alpha}{\cos\alpha + \cos 2\alpha + \cos 3\alpha + \ldots + \cos n\alpha}.$

4. $\tan n\alpha = \dfrac{\sin\alpha + \sin 3\alpha + \sin 5\alpha + \ldots + \sin(2n-1)\alpha}{\cos\alpha + \cos 3\alpha + \cos 5\alpha + \ldots + \cos(2n-1)\alpha}.$

5. $\tan\frac{1}{2}(n+1)(\pi + \alpha) = \dfrac{\sin\alpha - \sin 2\alpha + \sin 3\alpha - \ldots \text{ to } n \text{ terms}}{\cos\alpha - \cos 2\alpha + \cos 3\alpha - \ldots \text{ to } n \text{ terms}}.$

Sum to n terms the following series:

6. $\cos^2\alpha + \cos^2(\alpha + \beta) + \cos^2(\alpha + 2\beta) + \ldots.$

7. $\sin^3\alpha + \sin^3(\alpha + \beta) + \sin^3(\alpha + 2\beta) + \ldots.$

8. $\cos^4\alpha + \cos^4(\alpha + \beta) + \cos^4(\alpha + 2\beta) + \ldots.$

9. $\cos\theta\cos(\theta + \alpha) + \cos(\theta + \alpha)\cos(\theta + 2\alpha)$
$+ \cos(\theta + 2\alpha)\cos(\theta + 3\alpha) + \ldots.$

10. $\sin 2\theta\cos\theta + \sin 3\theta\cos 2\theta + \sin 4\theta\cos 3\theta + \ldots.$

11. $\sec\alpha\sec 2\alpha + \sec 2\alpha\sec 3\alpha + \sec 3\alpha\sec 4\alpha + \ldots$

12. $\dfrac{1}{\cos\alpha + \cos 3\alpha} + \dfrac{1}{\cos\alpha + \cos 5\alpha} + \dfrac{1}{\cos\alpha + \cos 7\alpha} + \ldots.$

13. $\sin^2\alpha\sin 2\alpha + \frac{1}{2}\sin^2 2\alpha\sin 4\alpha + \frac{1}{4}\sin^2 4\alpha\sin 8\alpha + \ldots.$

14. $\frac{\sin^3 a}{1} + \frac{\sin^3 3a}{3} + \frac{\sin^3 3^2 a}{3^2} + \ldots.$

15. $\frac{\cos^3 a}{1} - \frac{\cos^3 3a}{3} + \frac{\cos^3 3^2 a}{3^2} - \ldots.$

16. $\sin\theta \sin 3\theta + \sin 2\theta \sin 2 \,.\, 3\theta + \sin 2^2\theta \sin 2^2 \,.\, 3\theta + \ldots.$

17. $\cot\theta \cos\theta + 2 \cot 2\theta \cos 2\theta + 4 \cot 4\theta \cos 4\theta + \ldots.$

18. $\sec a \sin 2a \sec 3a + \sec 3a \sin 4a \sec 5a + \ldots.$

19. $\tan a + \cot a + \tan 2a + \cot 2a + \tan 4a + \cot 4a + \ldots.$

20. $\sin 3a \sec^2 a \sec^2 2a + \sin 5a \sec^2 2a \sec^2 3a + \ldots.$

21. $\sin\theta + 4 \sin 3\theta + 7 \sin 5\theta + 10 \sin 7\theta + \ldots.$

22. $\tan x \sec^2 x + \frac{1}{8} \tan\frac{x}{2} \sec^2\frac{x}{2} + \frac{1}{8^2} \tan\frac{x}{2^2} \sec^2\frac{x}{2^2} + \ldots.$

23. $\sin a \cos a + \sin 3a \cos 2a + \sin 5a \cos 3a + \ldots.$

24. $\sec a + \frac{1}{2} \sec a \sec 2a + \frac{1}{4} \sec a \sec 2a \sec 4a + \ldots.$

25. $\frac{1}{\cot a - 3\tan a} + \frac{3}{\cot 3a - 3\tan 3a} + \frac{3^2}{\cot 3^2 a - 3\tan 3^2 a} + \ldots$

26. $\frac{1}{3\cot a - \tan a} + \frac{3}{3\cot 3a - \tan 3a} + \frac{3^2}{3\cot 3^2 a - \tan 3^2 a} + \ldots$

27. $\sin a + x \sin 2a + x^2 \sin 3a + \ldots$

28. $\cos a + x \cos 2a + x^2 \cos 3a + \ldots.$

29. $\sin a + x \cos 2a - x^2 \sin 3a - x^3 \cos 4a + \ldots$

30. $\cos a + x \sin 2a - x^2 \cos 3a - x^3 \sin 4a + \ldots$

31. $\sin a + 2 \sin 2a + 3 \sin 3a + \ldots$

32. $\cos a + 2 \cos 2a + 3 \cos 3a + \ldots$

33. $\frac{1}{\cos^2\phi \cot\phi - \sin^2\phi \tan\phi} + \frac{1}{\cos^2 3\phi \cot\phi - \sin^2 3\phi \tan\phi} + \frac{1}{\cos^2 5\phi \cot\phi - \sin^2 5\phi \tan\phi} + \ldots$

34. $\tan\phi \sec 2\phi + \tan 2\phi \sec 4\phi + \tan 4\phi \sec 8\phi + \ldots$

CHAPTER XVIII.

ENDLESS SERIES.

§ 1. Convergency and Divergency.

Limits.

439. If two quantities x and y vary together in such a way that the difference of x from a may be made less than any assignable value by taking y near enough to b, while the difference of y from b may be made less than any assignable value by taking x near enough to a; then we say that **in the limit** $x=a$ when $y=b$, or $y=b$ when $x=a$.

If x and y so vary that x may be made greater than any assignable value by taking y near enough to b, while the difference of y from b may be made less than any assignable value by taking x large enough; then we say that **in the limit** $x=\infty$ when $y=b$, or $y=b$ when $x=\infty$.

If x and y so vary that either may be made greater than any assignable value by taking the other large enough; then we say that **in the limit** $x=\infty$ when $y=\infty$ or $y=\infty$ when $x=\infty$.

440. A sum of an endless number of terms can be used for arithmetical purposes, only if it is *convergent*.

Def. 1. An endless series is said to be **Convergent** if there is some *one* finite quantity towards which the sum of its terms approximates, in such wise that by increasing the number of these terms we may make the difference between their sum and the finite quantity less than any assignable value.

And this finite quantity is called the **Sum** of the endless series.

DEF. 2. An endless series is said to be **Divergent** if the sum of *any* number of its terms may be made greater than any assignable finite quantity by sufficiently increasing the number of terms.

441. According to the above definitions there are two kinds of series which are neither convergent nor divergent.

Thus a series which is such that, when $n = f(r)$ the sum tends towards some limit A when r is indefinitely increased, and when $n = \phi(r)$ the sum tends towards some other limit B when r is indefinitely increased, and so on: then the series is neither convergent nor divergent.

For example,

$$a - a + a - a + \ldots\ldots = 0 \text{ or } a,$$

according as $n = 2r$ or $2r + 1$.

$$1 - 1 + 2 - 2 + 3 - 3 + \ldots\ldots = 0 \text{ or } \infty,$$

according as $n = 2r$ or $2r + 1$.

If all the different limits are *finite*, the series may be called *Indeterminately finite:* if one of the limits is *infinite*, the series may be called *Indeterminately infinite.*

442. The above distinctions may be expressed by the use of the term *Limit;* thus

An endless series is convergent, if the sum of n terms, when $n = \infty$, has some single finite limit.

It is divergent, if, in the limit, its sum $= \infty$ when $n = \infty$.

It is indeterminately finite, if the sum of n terms, when $n = \infty$, has more than one limit which is always finite.

It is indeterminately infinite, if the sum of n terms, when $n = \infty$, has infinite and finite limits.

443. Since the nth term of a series is equal to the difference between the sum of n terms and of $n - 1$ terms, *a series cannot be convergent unless, when n is infinite, the limit of the nth term is zero.* This is a *necessary* but not a *sufficient* condition for convergency.

If the limit of the nth term is *not zero, but finite*, the sum of the series may be *finite* for *all* values of n however great, but it cannot have a *single limit*. Thus the series

$$-1+\tfrac{3}{2}-\tfrac{5}{3}+\tfrac{7}{4}-\ldots+(-1)^n\frac{2n-1}{n}+\ldots$$

is *finite*, but it has a different limit according as n is *even* or *odd*.

If the limit of the nth term is *infinite*, the sum of the series may be *finite* for *some* values of n however great, but it cannot be finite for *all* values of n. Thus the series

$$-0+\tfrac{1}{2}-\tfrac{5}{3}+\tfrac{7}{4}-\tfrac{14}{5}+\tfrac{17}{6}-\ldots-\frac{2n^2-n-1}{2n-1}+\frac{2n^2-1}{2n}+\ldots$$

is finite for *even* values, but infinite for odd values of the number of terms taken.

444. A series, *all of whose terms have the same sign*, must be either convergent or divergent.

For if, for *some* forms of f and ϕ, the sum of $f(r)$ terms differs from that of $\phi(r)$ terms by a quantity whose limit—when r is ∞—is *not* zero, the series must be infinite for *all* values of n, when n is infinite. Compare Art. 449.

But if, for *all* forms of f and ϕ, the sum of $f(r)$ terms differs from that of $\phi(r)$ terms by a quantity whose limit—when r is ∞—*is* zero, the series cannot have *more than one* finite limit.

445. If a series, *all of whose terms have the same sign*, is convergent, the series obtained from it by changing the sign of any of the terms will be convergent.

For the new series cannot have a larger arithmetical value than the old. And if, in the new series the difference between $f(r)$ terms and $\phi(r)$ terms—when r is ∞—were not zero, it could not be zero in the old: hence, as shown in the last article, the old series would be divergent, which is contrary to the hypothesis.

Similarly, if a series, *some of whose terms are different in sign*, is divergent, the series obtained by making the sign of every term the same will be divergent.

Theorems on Alternately signed series.

446. (I.) *A series which is arranged in groups alternately signed will be finite for all values of* n *if every group is arithmetically equal to or less than the preceding.*

Let $$u_1 - u_2 + u_3 - u_4 + \dots$$

be the given series, where u_n is always positive and = or $> u_{n+1}$.

The sum

$= (u_1 - u_2) + (u_3 - u_4) + \dots$ and is therefore *positive:*

and $= u_1 - (u_2 - u_3) - (u_4 - u_5) - \dots$ and is therefore *algebraically less than* u_1.

Hence the series must be finite for all values of n.

If, however, the limit of the nth group is not zero, the sum of n groups will always differ from that of $n-1$ groups, when n is *infinite.*

But we see by the first of the above forms of bracketing, that the series is equivalent to a series all of whose terms are *positive.* Hence, by Art. 444, if the limit of the nth term *is* zero, there can be only one limit of the sum; hence the series is *convergent.*

447. (II.) Hence, *a series which is arranged in groups alternately signed will be finite for all values of* n, *if every group is arithmetically greater than the preceding but less than some finite quantity.*

For let every group be less than the finite quantity a. Then

$$-u_1 + u_2 - u_3 + \dots = -(a - v_1) + (a - v_2) - (a - v_3) + \dots$$
$$= v_1 - v_2 + v_3 - \dots$$

where v_1, v_2, v_3... are continually decreasing. Hence, by the last theorem, the series is finite for all values of n.

448. (III.) *A series, which is arranged in groups alternately signed, is divergent, if the ratio of every group to the preceding is greater than some quantity greater than* 1.

For let $\frac{u_{n+1}}{u_n} > k$, where $k > 1$;

Then $u_{n+1} > k^n u_1$; which can be made greater than any assignable quantity by increasing n.

$\therefore$ even if the sum of n terms were finite, the sum of $n+1$ terms would be infinite.

Also $-u_{n-1} + u_n > (k-1)\, u_{n-1} = \infty$ when n is ∞.

Hence, into whatever pairs we group the series, it is always *infinite.*

Thus, the series is *divergent.*

Theorems on Series of same sign throughout.

449. (I.) *A series, all of whose terms have the same sign, is* divergent, *if every term is* equal to or greater than *the preceding.*

For in this case the sum of n terms of

$$u_1 + u_2 + \ldots + u_n$$

is equal to or greater than $n \,.\, u_1$; and is, therefore, infinite when n is infinite.

The typical series here is that in which the general or nth term is x^n where x is $=$ or > 1.

450. (II.) *A series, all of whose terms have the same sign, is* convergent, *if the ratio of every term to the preceding is* less than some quantity less than unity.

Let $\frac{u_{n+1}}{u_n} < k$ where $k < 1$. Then

$$(1-k)\,(u_1 + u_2 + u_3 + \ldots) = u_1 - ku_1 + u_2 - ku_2 + \ldots$$

a finite series, by Art. 446, since $u_{n+1} < ku_n$, and $ku_n < u_n$.

Hence $u_1 + u_2 + u_3 + \ldots$ is *finite,* and $\therefore$ *convergent.*

The typical series here is that in which the general or nth term is x^n, where x is < 1.

The preceding two theorems say nothing of the case in which the ratio of each term to the preceding is less than unity, *but approaches unity by a quantity whose limit is zero.*

451. (III.) *A series, all of whose terms have the same sign, is* divergent, *if every term multiplied by its 'serial number' is* equal to or greater than *the preceding term multiplied by its 'serial number.'*

$$\text{Let } (n+1)\,u_{n+1} = nu_n = \ldots = 2u_2 = u_1.$$

Then, observing that every term is less than the preceding but approaches indefinitely near to it, we see that

$$u_3 + u_4 > 2u_4, \text{ i.e. } > u_2,$$
$$u_5 + u_6 + u_7 + u_8 > 4u_8, \text{ i.e. } > u_2,$$

and so on.

Hence, by (I), the series is infinite.

A fortiori, if $(n+1)\,u_{n+1} > nu_n$, the series is *divergent.*

These conditions may be expressed as follows:

$$\frac{u_{n+1}}{n\,(u_n - u_{n+1})} \gtreqless 1 \text{ or } n\left(\frac{u_n}{u_{n+1}} - 1\right) \gtreqless 1.$$

The typical series here is that, in which the general or nth term is n^x where $x = \text{or} > -1$.

Thus the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots$ is divergent.

452. (IV.) *A series, all of whose terms have the same sign, is* convergent, *if throughout*

$$\frac{u_{n+1}}{n\,(u_n - u_{n+1})} < k, \text{ where } k < 1.$$

For here

$$(1-k)\,(u_1 + u_2 + u_3 + \ldots + u_n)$$
$$= u_1 + u_2 + \ldots + u_n$$
$$- k\,\{(u_1 - u_2) + 2\,(u_2 - u_3) + \ldots + (n-1)\,(u_{n-1} - u_n) + nu_n\}.$$

This is clearly less than u_1: for $u_2 < k\,(u_1 - u_2)$, $u_3 < 2k\,(u_2 - u_3)$, and so on.

Hence the series $u_1 + u_2 + u_3 + \ldots$ is convergent.

This condition may be written in the form

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) > k, \text{ where } k > 1.$$

Example. The series in which the ratio of the $(n+1)$th to the nth term is $1 - \frac{x}{n}$ is convergent, if $x > 1$. For here

$$\frac{u_{n+1}/u_n}{n\,(1 - u_{n+1}/u_n)} = \frac{1}{x} - \frac{1}{n},$$

which is always less than a quantity $\frac{1}{x}$ which is less than 1.

453. (V.) *If* $u_1 + u_2 + u_3 + \ldots$ *and* $v_1 + v_2 + v_3 + \ldots$ *in each of which all the terms have the same sign be two series such that the limit of* $u_n : v_n$ *and of* $v_n : u_n$ *is finite, then both series are convergent or both series are divergent.*

For since the limit of $u_n : v_n$ is finite, the value of $u_n : v_n$ must always be less than some finite quantity k say.

Thus $u_1 < kv_1$; $u_2 < kv_2$, &c.

$$\therefore u_1 + u_2 + \ldots + u_n + \ldots < k\,(v_1 + v_2 + \ldots + v_n + \ldots).$$

$\therefore$ If the v-series is finite, the u-series is finite;

if the u-series is infinite, the v-series is infinite.

In the same way, since the limit of $v_n : u_n$ is finite, we may show that

If the u-series is finite, the v-series is finite;

if the v-series is infinite, the u-series is infinite.

454. (VI.) *If* $f(n)$ *always decreases as* n *increases and is always positive, then according as the series whose general term is* $f(n)$ *is convergent or divergent so also is the series whose general term is* $\frac{f(\log_b n)}{n}$, *where* b *is greater than* 1.

Since $b > 1$, $\therefore$ as n increases, $\log_b n$ increases, and $\therefore f(\log_b n)$ decreases.

(1) Take b to be integral.

Consider the terms from $\frac{f(\log_b b^n)}{b^n}$ to $\frac{f(\log_b b^{n+1})}{b^{n+1}}$, excluding the last. The number of these terms is $b^{n+1} - b^n$.

Hence their sum

$$< (b^{n+1} - b^n) \frac{f(\log_b b^n)}{b^n}, \text{ i.e. } < (b-1) f(n),$$

and $$> (b^{n+1} - b^n) \frac{f(\log_b b^{n+1})}{b^{n+1}}, \text{ i.e. } > (1 - 1/b) f(n+1).$$

Thus, the ratio of the series $\frac{f(\log_b n)}{n}$ to the series $f(n)$ lies between $1 - 1/b$ and $b - 1$. Hence the two series are either both convergent or both divergent.

(2) Take b to be fractional, and to lie between the two integers i and j.

Then $\log_b n$ lies between $\log_i n$ and $\log_j n$.

Hence, for *all* values of $b > 1$, the series $\frac{f(\log_b n)}{n}$ is convergent or divergent according as the series $f(n)$ is convergent or divergent.

455. Now the series x^n is convergent or divergent according as x is less or not less than 1.

$\therefore$ the series

$$\frac{x^{\log_b n}}{n} \text{ i.e. } \frac{n^{\log_b x}}{n} \text{ i.e. } n^{\log_b x - 1}$$

is convergent or divergent according as x is less or not less than 1: i.e. according as $\log_b x$ is less or not less than 0.

Thus *the series n^x is convergent or divergent according as x is less or not less than* -1.

The case of divergency here was proved more directly in Art. 451.

Cor. Hence all the series

$$\frac{(\log n)^x}{n}, \frac{\{\log(\log n)\}^x}{n \log n}, \frac{[\log\{\log(\log n)\}]^x}{n \log n \log(\log n)}, \text{ \&c. \&c.,}$$

are convergent or divergent according as x is less or not less than -1.

456. By means of the comparative theorem V, we may determine the character of many series.

Thus if the nth term of a series can be expressed as a finite or convergent sum of terms involving *descending* powers of n, the series will be convergent or divergent according as the highest index of n is or is not less than -1.

For the nth term $an^x + bn^y + \ldots$ where the indices are descending has a finite ratio to the nth term n^x.

457. It should be noticed that the assigned characteristics in the above theorems may be supposed to begin in a given series at any term *at a finite distance after the first.* Thus u_1 in the above theorems means 'that term at which the assigned characteristic begins to hold'.

Principle of Equating Coefficients.

458. *If an endless series in powers of* x *is equal to* zero, *for* every *value of* x *from* 0 *to some finite value* f, *then* each coefficient *must separately be zero.*

For let a_0, the coefficient of x^r, be the first coefficient which is supposed not to be zero.

Then, for the values of x in question,

$$a_0x^r + a_1x^{r+1} + a_2x^{r+2} + \ldots = 0,$$

$$\text{i.e. } x^{r+1}(a_1 + a_2x + \ldots) = -a_0x^r.$$

If then x is not equal to zero, dividing by x^r,

$$x(a_1 + a_2x + \ldots) = -a_0, \text{ a finite constant.}$$

$\therefore$ when $x = f$, the series within the bracket is *finite*; and it cannot become infinite by decreasing x.

But, if a_0 is not zero, this series would have to become greater than any assignable quantity when x becomes sufficiently small. But this has been shown to be impossible. Hence a_0 must be zero; and, therefore, every coefficient must be zero.

459. Hence, *if two endless series in powers of* x *are equal to one another for every value of* x *between* 0 *and* f, *then the corresponding coefficients in the two series must be severally equal.*

In the above it is essential to note that the two series must be *convergent*, otherwise we cannot assign any value to their sum, nor therefore speak of them as being equal.

The proof follows immediately from the preceding article. For,

If $a_0 + a_1x + a_2x^2 + \ldots = b_0 + b_1x + b_2x^2 + \ldots$

then $a_0 - b_0 + (a_1 - b_1)\,x + (a_2 - b_2)\,x^2 + \ldots = 0.$

$\therefore$ by above,

$$a_0 - b_0 = a_1 - b_1 = a_2 - b_2 = \ldots = 0,$$

$\therefore$ $a_0 = b_0,\ a_1 = b_1,\ a_2 = b_2 \ldots.$

Convergency of the Binomial Series.

460. Consider the series

$$a^m + ma^{m-1}x + \frac{m\,(m-1)}{1\,.\,2}\,a^{m-2}x^2 + \ldots + \frac{m\,(m-1)\ldots(m-r+1)}{\lfloor r}\,a^{m-r}x^r + \ldots$$

where m is *finite*.

Here the multiplier $\dfrac{u_{r+1}}{u_r} = \dfrac{m-r+1}{r}\,.\,\dfrac{x}{a} = -\dfrac{x}{a}\left(1 - \dfrac{m+1}{r}\right).$

We will take r to be *algebraically* $> m+1$, so that the factor in the bracket is positive and the multiplier has the sign of $-x/a$.

A. *Let* $m+1 > 0$; then the multiplier is always arithmetically $< x/a$; but increases up to the limit x/a, as r increases.

Hence, if x/a is arithmetically < 1, the multiplier is less than a quantity less than 1; $\therefore$ the series is *convergent*.

If x/a is arithmetically > 1, then, when r is sufficiently large, the multiplier becomes greater than a quantity greater than 1; $\therefore$ the series is *divergent*.

B. *Let* $m+1 < 0$; then the multiplier is always arithmetically $> x/a$; but decreases down to the limit x/a, as r increases.

Hence, if x/a is arithmetically < 1, then, when r is sufficiently large, the multiplier becomes less than a quantity less than 1; $\therefore$ the series is *convergent.*

If x/a is arithmetically > 1, the multiplier is greater than a quantity greater than 1; $\therefore$ the series is *divergent.*

Hence, *if* x/a *is not* $+$ *or* -1, *the series is convergent or divergent according as* x/a *is arithmetically less or greater than* 1.

461. *Let* $x/a = -1$; then the multiplier is positive when r is algebraically $> m+1$, so that all the terms have the same sign, and the series is either convergent or divergent.

A. Let $m+1<0$; then the multiplier is greater than 1, and $\therefore$ the series is *divergent.*

B. *Let* $m+1>0$; then the multiplier is less than 1, but increases up to the limit 1. In this case

$$\frac{u_{r+1}/u_r}{r\,(1-u_{r+1}/u_r)} = \frac{1-(m+1)/r}{m+1} = \frac{1}{m+1} - \frac{1}{r}.$$

$\therefore$ if $m+1>0$ but <1, this expression, when r is sufficiently large, is greater than 1; $\therefore$ the series is *divergent.*

But if $m+1>1$, this expression is less than a quantity less than 1, $\therefore$ the series is *convergent.*

Thus, *when* x/a $= -1$, *the series is convergent or divergent according as* m *is positive or negative.*

462. *Let* $x/a = +1$; then the multiplier is negative when r is algebraically $> m+1$, so that the terms are alternately signed.

A. *Let* $m+1>0$; then the multiplier is arithmetically less than 1; $\therefore$ the terms are arithmetically decreasing, and the series is *finite* for all values of n.

Moreover $$-\frac{u_r}{u_{r+1}} = \frac{r}{r-m-1} = 1 + \frac{m+1}{r-m-1},$$

$$\therefore \quad \pm\frac{u_r}{u_{n+1}} = \left(1+\frac{m+1}{r-m-1}\right)\left(1+\frac{m+1}{r-m}\right)\cdots\left(1+\frac{m+1}{n-m-1}\right)$$

$$> 1 + \frac{m+1}{r-m-1} + \frac{m+1}{r-m} + \ldots + \frac{m+1}{n-m-1}.$$

Now, if n is indefinitely increased, the terms of this series have a finite ratio to the series

$$\frac{1}{r}+\frac{1}{r+1}+\ldots+\frac{1}{n}+\ldots$$

Hence this series is *infinite*. $\therefore \dfrac{u_r}{u_{n+1}}=\infty$ when $n=\infty$.

But u_r is *finite*, $\therefore$ u_{n+1} is 0.

Thus the limit, when $n=\infty$, of the n^{th} term of the original binomial series is zero: $\therefore$ the series is *convergent*.

B. *Let* $m+1<0$: then the multiplier is arithmetically greater than 1, but decreases down to the limit 1.

Here $$\frac{u_{r+1}}{-u_r}=1-\frac{m+1}{r},$$

$$\therefore \quad \pm\frac{u_{n+1}}{u_r}=\left(1-\frac{m+1}{r}\right)\left(1-\frac{m+1}{r+1}\right)\ldots\left(1-\frac{m+1}{n}\right)$$

$$>1-\frac{m+1}{r}-\frac{m+1}{r+1}-\ldots-\frac{m+1}{n}.$$

Now, if n is indefinitely increased, this series is infinite; $\therefore$ u_{n+1} is *infinite*. Hence for *some* values of $n=\infty$, the binomial series is infinite.

Moreover, taking the terms in *pairs*, we have

$$\frac{u_{r+1}+u_r}{u_{r-1}+u_{r-2}}=\frac{(r-m-2)\,(r-m-3)}{r\,(r-1)},$$

which is >1 if $m+2$ is negative; but, if not,

$$\frac{r}{2}\left(\frac{u_{r-1}+u_{r-2}}{u_{r+1}+u_r}-1\right)=\frac{(m+2)\{1-(m+3)/2r\}}{\{1-(m+2)/r\}\{1-(m+3)/r\}}.$$

This expression $=m+2$ nearly, if r is large enough. But $m+2<1$, since $m+1<0$.

Hence, by Art. 451, the series is infinite for *all* values of n, and is therefore divergent.

Thus, *when* $\mathrm{x/a}=1$, *the series is convergent or divergent according as* m *is greater or less than* -1.

§ 2. The Binomial Theorem.

The index theorem.

463. *If, for all positive and negative values of m and n, $f(m)$ and $f(n)$ are single-valued functions of m and n such that $f(m) \times f(n) = f(m+n)$, then, for all such values, $f(n)$ is one of the values of $\{f(1)\}^n$.*

(1) *Let n be a positive integer.*

Then $f(1) \times f(1) \times \ldots$ to n factors $= f(1 + 1 + \ldots$ to n terms),

i.e. $\{f(1)\}^n = f(n)$.

(2) *Let $n = 0$.*

Then $f(n) \times f(0) = f(n+0) = f(n)$.

$\therefore f(0) = 1 = \{f(1)\}^0$.

(3) *Let n be a negative integer $= -m$ say.*

Then $f(m) \times f(-m) = f(m-m) = f(0) = 1$ by (2).

$$\therefore f(-m) = \frac{1}{f(m)} = \frac{1}{\{f(1)\}^m} \text{ by (1)},$$
$$= \{f(1)\}^{-m}.$$

(4) *Let n be fractional $= \frac{p}{q}$ say,* where p and q are integral, and q positive.

Then

$$f\left(\frac{p}{q}\right) \times f\left(\frac{p}{q}\right) \times \ldots \text{ to } q \text{ factors} = f\left(\frac{p}{q} + \frac{p}{q} + \ldots \text{ to } q \text{ terms}\right),$$

$$\text{i.e. } \left\{f\left(\frac{p}{q}\right)\right\}^q = f(p) = \{f(1)\}^p \text{ by (1) and (3).}$$

$\therefore f\left(\frac{p}{q}\right)$ is one of the values of $\sqrt[q]{\{f(1)\}^p}$, i.e. of $\{f(1)\}^{\frac{p}{q}}$.

464. When m is a positive integer, it is easily proved by the theory of combinations that

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{1 \cdot 2} x^2 + \ldots + \frac{m(m-1)\ldots(m-r+1)}{\lfloor r} x^r + \ldots$$

Now the series on the right side is endless unless m *is* a positive integer. It is natural to enquire, then, what is the value of this endless series, when m is other than a positive integer, and when m and x have such values as will secure its convergency, as investigated in Arts. 460—462.

Now multiplying the above series by $1+x$, the coefficient of x^r in the product is

$$\frac{m(m-1)\dots(m-r+2)}{\lfloor r-1}\left(1+\frac{m-r+1}{r}\right)=\frac{(m+1)m(m-1)\dots(m-r+2)}{\lfloor r}.$$

Hence calling the series $f(m)$, we have

$$f(m)\times(1+x)=f(m+1).$$

Multiplying by $1+x$, r times, we have

$$f(m)\times(1+x)^r=f(m+r).$$

Now $f(0)=1$; $\therefore$ putting $m=0$ in above,

$$(1+x)^r=f(r)\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\text{(I)}.$$

Putting $m=-r$,

$$f(-r)\times(1+x)^r=f(0)=1,$$

$$\therefore f(-r)=(1+x)^{-r}\dots\dots\dots\dots\dots\dots\text{(II)}.$$

Hence for all positive or negative *integral* values of m,

$$f(m)=(1+x)^m.$$

By means of the *index theorem* we will find the value of $f(m)$ for any positive or negative value whatever.

465. *Lemma.* If two sets of quantities S_0, S_1, $\dots S_r$ and T_0, T_1, $\dots T_{r+1}$ are so related that, for any suffix p,

$$(m-r+p)\,S_p=(r-p+1)\,T_p \text{ and } (n-p+1)\,S_{p-1}=pT_p;$$

then

$(m+n-r)$ (*sum of the* S's) will equal $(r+1)$ (*sum of the* T's).

In the first of the given equations, put $p=0$:

$$\text{Thus } (m-r)\,S_0=(r+1)\,T_0.$$

In the last of the given equations, put $p=r+1$:

$$\text{Thus } (n-r)\,S_r=(r+1)\,T_{r+1}.$$

Thus

$$(m+n-r)\,S_0=(m-r)\,S_0+nS_0=(r+1)\,T_0+\ldots$$

and so on.

$$(m+n-r)\,S_{p-1}=(m-r+p-1)\,S_{p-1}+(n-p+1)\,S_{p-1}=\ldots+pT_p,$$

$$(m+n-r)\,S_p=(m-r+p)\,S_p+(n-p)\,S_p=(r-p+1)\,T_p+\ldots,$$

and so on.

$$(m+n-r)\,S_r=mS_r+(n-r)\,S_r=\ldots+(r+1)\,T_{r+1}.$$

Adding all these equations together, we see that the coefficient of T_p is $r+1$ for every value of p.

Hence

$$(m+n-r)\ (\textit{sum of the } S\text{'s})=(r+1)\ (\textit{sum of the } T\text{'s}) \qquad \text{Q.E.D.}$$

466. *To prove universally that the series*

$$1+mx+\frac{m\,(m-1)}{1\,.\,2}x^2+\ldots+\frac{m\,(m-1)\ldots(m-r+1)}{\lfloor r}x^r+\ldots$$

for all positive or negative values of m *for which it is convergent, is* one *of the values of* $(1+x)^m$.

Here put

$$f(m)\equiv 1+mx+\frac{m\,(m-1)}{1\,.\,2}x^2+\ldots+\frac{m\,(m-1)\ldots(m-r+1)}{\lfloor r}x^r+\ldots$$

$$f\,(n)\equiv 1+nx+\frac{n\,(n-1)}{1\,.\,2}x^2+\ldots+\frac{n\,(n-1)\ldots(n-r+1)}{\lfloor r}x^r+\ldots$$

In the above lemma,

let $T_0=\dfrac{m\,(m-1)\ldots(m-r)}{\lfloor r+1}$, then $S_0=\dfrac{m\,(m-1)\ldots(m-r+1)}{\lfloor r}$,

$$T_1=\frac{m\ldots(m-r+1)}{\lfloor r}\cdot\frac{n}{1}, \qquad S_1=\frac{m\ldots(m-r+2)}{\lfloor r-1}\cdot\frac{n}{1},$$

$$T_2=\frac{m\ldots(m-r+2)}{\lfloor r-1}\cdot\frac{n\,(n-1)}{1\,.\,2}, \qquad S_2=\frac{m\ldots(m-r+3)}{\lfloor r-2}\cdot\frac{n\,(n-1)}{1\,.\,2},$$

and so on.

Hence, multiplying the two series $f(m), f(n)$,

coeff. of x^r in $f(m) \times f(n) = S_0 + S_1 + S_2 + \ldots + S_r$,

coeff. of x^{r+1} in $f(m) \times f(n) = T_0 + T_1 + T_2 + \ldots + T_r + T_{r+1}$,

$\therefore$ by lemma, coeff. of $x^{r+1} =$ (coeff. of x^r) $\times \dfrac{m+n-r}{r+1}$.

Now coeff. of $x^0 = 1$. Hence putting $r = 0, 1, 2 \ldots$ successively. coeff. of

$$x^{r+1} \text{ in } f(m) \times f(n) = \frac{(m+n)(m+n-1)\ldots(m+n-r)}{\underline{|r+1}},$$

$\therefore$
$$f(m) \times f(n) = f(m+n).$$

Hence, for all real values of m, $f(m)$ is one value of $\{f(1)\}^m$. Now $f(1) = 1 + x$. $\therefore$ the series $f(m)$ is one value of $(1+x)^m$.

467. The above series, which involves m and x, may be called $f(m, x)$.

If now m is *integral*, $(1+x)^m$ has but one value. Hence we may say $f(m, x) = (1+x)^m$.

But if m is *fractional* = say p/q, $(1+x)^m$ has a number (at present unknown) of values. It has been proved, in Art. 230, however, that the q^{th} root of a *positive* quantity has only one *positive* value.

Now for all values of $f(m, x)$ which are convergent, x is numerically not greater than 1, so that for all such values $1 + x$ is *positive*. We will proceed to show that for all positive or negative values of m and x, which make $f(m, x)$ endless but convergent, $f(m, x)$ is *positive*.

(I) *Let* x *and* m *be both negative.*

Then every term in $f(m, x)$ is positive: $\therefore f(m, x)$ is positive.

(II) *Let* x *be negative,* and m *positive.*

Then
$$f(m, x) = f(n, x) \times f(m-n, x).$$

Suppose n is any integer greater than m.

Then $f(n, x) = (1+x)^n$ and is $\therefore$ positive.

And $f(m-n, x)$ is positive by (I).

$\therefore$ also $f(m, x)$ is positive.

(III) *Let* x *be positive, and* m + 1 *positive.*

Then all the terms in $f(m, x)$ up to x^r, where r is not $> m+1$, are *positive*, after which they are alternate in sign but numerically decreasing; $\therefore f(m, x)$ is positive.

(IV) *Let* x *be positive, and* m + 1 *negative.*

Then $$f(m, x) = f(m+n, x) \div f(n, x).$$

Suppose that n is an integer, such that $m+n+1$ is *positive.*

Then $f(n, x) = (1+x)^n$ and is $\therefore$ positive.

And $f(m+n, x)$ is positive by (III).

$\therefore$ also $f(m, x)$ is positive.

Since then $f(m, x)$ is always *positive*, when it is endless but convergent, (m and x being real),

When m is *integral* $f(m, x) = (1+x)^m$.

When m is *fractional*, $f(m, x) =$ the one positive value of $(1+x)^m$.

§ 3. Exponential and Logarithmic Series.

468. *To find the limit of the product of an infinite number of factors, the limit of each of which is unity.*

A. Let *each* factor reach its limit unity, independently of the indefinite increase in the *number* of factors.

Then, call the first n factors $1+\rho_1$, $1+\rho_2$, $\ldots 1+\rho_n$; where $\rho_1, \rho_2, \ldots \rho_n$ are ultimately zero.

Assume the theorem that for n factors

$$(1+\rho_1)(1+\rho_2)\ldots(1+\rho_n) = 1+R,$$

where R is ultimately zero.

Introduce another factor $1+\rho_{n+1}$; then

$$(1+\rho_1)(1+\rho_2)\ldots(1+\rho_{n+1}) = 1+\rho_{n+1}+R+\rho_{n+1}R = 1+R',$$

where R' is ultimately zero.

Hence, if the theorem is true for n factors, it is true for $n+1$. Hence it is true, *however large the number of factors may be.*

Observe here particularly that we have allowed each factor to reach its limiting value, *before introducing the next factor.* This was legitimate because of the special hypothesis (A).

B. Let each factor only reach its limiting value, *when the number of factors is indefinitely increased.* In other words, *let each factor be a function of the number of factors.* Then the above argument will not hold. An important example is investigated in Art. 469.

Similar propositions hold with respect to the *sum* of an infinite number of terms, the limit of *each* of which is *zero.*

If *each* term reaches its limit zero independently of the indefinite increase in the number of terms, the limit of the sum is *zero.* But if each term is a function of the number of terms, this is not necessarily the case.

These cases of multiplication and summation are particular cases of the evaluation of expressions of the form 1^{∞} and $0 \times \infty$.

Now it is clear that however many times we multiply 1 by itself the result is 1; and however many times we add 0 to itself, the result is 0.

Hence, when the symbols represent mere independent numbers, we have $1^{\infty} = 1$ and $0 \times \infty = 0$.

But, when the symbols represent the limiting values of *co-varying* quantities, these results are not necessarily true: and the values cannot be determined without assigning what function the one quantity is of the other.

The student may be warned against confounding the products or series here discussed, with those in which the limit of the n^{th} factor or term, not of *each* factor or term, is unity or zero.

469. *To find an expression for the limit of* $\left(1 + \frac{1}{n}\right)^{nx}$, *when* n *is indefinitely increased.*

[Here $1 + 1/n$ is a factor whose limit is unity and which has to be multiplied by itself an infinite number of times. But each factor only reaches its limit, as the number of factors is indefinitely increased. Hence we cannot employ the argument of Art. 468 (A).]

For all *finite* values of n greater than 1, the positive value of $\left(1+\frac{1}{n}\right)^{nx}$

$$=1+nx.\frac{1}{n}+\frac{nx(nx-1)}{1.2}.\left(\frac{1}{n}\right)^2+\ldots+\frac{nx(nx-1)\ldots(nx-r+1)}{\lfloor r}\left(\frac{1}{n}\right)^r+\ldots$$

$$=1+x+\frac{x^2}{\lfloor 2}\left(1-\frac{1}{nx}\right)+\ldots+\frac{x^r}{\lfloor r}\left(1-\frac{1}{nx}\right)\left(1-\frac{2}{nx}\right)\ldots\left(1-\frac{r-1}{nx}\right)+\ldots$$

In this series, the multiplier

$$\frac{u_{r+1}}{u_r}=\left(1-\frac{r-1}{nx}\right)\frac{x}{r}=\frac{x}{r}-\frac{1}{n}+\frac{1}{rn}.$$

This decreases as r increases, and is less than a quantity less than 1, if r is large enough. This is true, even if n is infinite. Hence the series is convergent for all values of n however great.

Now when n is indefinitely increased, $\frac{1}{nx}$ is indefinitely small; $\therefore \frac{r-1}{nx}$ is indefinitely small, until r becomes commensurate with n; since the term $\frac{1}{nx}$ decreases independently of the increase in r.

Hence, the limit of each of the factors

$$\left(1-\frac{1}{nx}\right),\ \left(1-\frac{2}{nx}\right),\ldots\left(1-\frac{r-1}{nx}\right)$$

is unity. And they reach their limit independently of the increase in their number r; $\therefore$ their product is unity.

When r becomes commensurate with n, the sum of the terms beginning with x^r is indefinitely small (because the series is convergent): hence no error arises from falsely evaluating this sum.

Thus the limit when n is indefinitely increased of the positive value of

$$\left(1+\frac{1}{n}\right)^{nx} \text{ is } 1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\ldots+\frac{x^r}{\lfloor r}+\ldots$$

for all real finite values of x.

470. In the above result, put $x = 1$. Thus the limit of

$$\left(1+\frac{1}{n}\right)^n \text{ is } 1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\ldots+\frac{1}{\lfloor r}+\ldots\ldots\ldots\ldots(1).$$

Again, since $\left\{\left(1+\frac{1}{n}\right)^n\right\}^x = \left(1+\frac{1}{n}\right)^{nx}$, we have

$$\left\{1+1+\frac{1}{\lfloor 2}+\ldots+\frac{1}{\lfloor r}+\ldots\right\}^x = 1+x+\frac{x^2}{\lfloor 2}+\ldots+\frac{x^r}{\lfloor r}+\ldots\ldots\ldots(2).$$

This important formula is called the **Exponential Theorem.**

Example. Prove (2) directly by means of the index theorem.

If $f(x)$ represents the right hand of (2); the terms of the r^{th} dimension in $f(x) \times f(y)$, are clearly

$$\frac{1}{\lfloor r}\left[x^r + r.x^{r-1}y + \frac{r(r-1)}{\lfloor 2}x^{r-2}y^2 + \frac{r(r-1)(r-2)}{\lfloor 3}x^{r-3}y^3+\ldots\right]$$

$$= \frac{(x+y)^r}{\lfloor r} \text{ by the Binomial Theorem for a positive integer.}$$

$\therefore f(x)\times f(y) = f(x+y).$

$\therefore f(x) = \{f(1)\}^x.$ Q. E. D.

471. The limit, when n is infinite, of $\left(1+\frac{1}{n}\right)^n$ is usually denoted by the symbol e (or ϵ).

Thus (1) and (2) of the last article become

$$e = 1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\ldots+\frac{1}{\lfloor r}+\ldots\ldots\ldots\ldots\ldots(3),$$

$$e^x = 1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\ldots+\frac{x^r}{\lfloor r}+\ldots\ldots\ldots\ldots\ldots(4).$$

Example. Find the value of $\frac{1}{2}(e^x+e^{-x})$ and $\frac{1}{2}(e^x-e^{-x})$.

$$e^{-x} = 1-x+\frac{x^2}{\lfloor 2}-\frac{x^3}{\lfloor 3}+\ldots\ldots$$

$$\therefore \tfrac{1}{2}(e^x+e^{-x}) = 1+\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}+\ldots\ldots$$

and $$\tfrac{1}{2}(e^x-e^{-x}) = x+\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}+\ldots\ldots$$

472. *To find the limits within which the value of* e *lies.*

Take each of the terms in (3) expressed decimally and found by dividing the preceding by 2, 3, 4, &c.: and then add up the terms as far as calculated. Thus

$$e > 2;\ > 2{\cdot}5;\ > 2{\cdot}6\dot{};\ > 2{\cdot}708\dot{3};\ > 2{\cdot}716\dot{6}.$$

On the other hand,

$$e < 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^r} + \ldots\right) < 1 + \frac{1}{1 - \frac{1}{2}} < 3.$$

$$e < 1 + 1 + \frac{1}{2}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\right) < 2 + \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{3}} < 2{\cdot}75.$$

$$e < 1 + 1 + \frac{1}{2} + \frac{1}{6}\left(1 + \frac{1}{4} + \ldots\right) < 2{\cdot}5 + \frac{1}{6} \cdot \frac{1}{1 - \frac{1}{4}} < 2{\cdot}72\dot{}.$$

Its value may be easily approximated to 2·71828182.

Logarithms to base e.

473. The expansion

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \ldots + \frac{x^r}{\lfloor r} + \ldots \quad \ldots\ldots\ldots\ldots\ldots(4),$$

enables us to find logarithms to base e as follows.

Let $x = y \log_e a$. Then $e^x = (e^{\log_e a})^y = a^y$,

$$\therefore \qquad a^y = 1 + y \log_e a + \frac{(y \log_e a)^2}{\lfloor 2} + \ldots + \frac{(y \log_e a)^r}{\lfloor r} + \ldots \quad \ldots(5).$$

The series (4) holds for *all* finite values of x. Hence the series (5) holds for all finite values of y and a, provided that a is positive, so that $\log_e a$ has a single real value positive or negative. (See Art. 243.)

Now let $a = z + 1$, *where* z *is any quantity numerically not greater than* 1: so that a is positive as required.

Also, by the binomial theorem, if y is positive, [See Arts. 461, 462.]

$$(1 + z)^y = 1 + yz + \frac{y(y-1)}{1 \cdot 2} z^2 + \ldots + \frac{y(y-1)\ldots(y-r+1)}{\lfloor r} z^r + \ldots \quad (6),$$

for any value of z numerically not greater than 1.

Hence the series (5) and (6) are equal for every value of y between 0 and any positive finite quantity.

Hence we may equate the coefficients of the corresponding powers of y in the two series.

Equating the coefficient of y,

$$\log_e(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots + (-1)^{r-1}\cdot\frac{z^r}{r} + \ldots \quad \ldots\ldots(7),$$

for all values of z, numerically not greater than 1.

Example. Expand $\log_e \sec\theta$ in powers of $\tan\theta$, if $\tan\theta < 1$.

$$\log_e \sec\theta = \tfrac{1}{2}\log_e \sec^2\theta = \tfrac{1}{2}\log_e(1+\tan^2\theta) = \tfrac{1}{2}\tan^2\theta - \tfrac{1}{4}\tan^4\theta + \tfrac{1}{6}\tan^6\theta - \ldots$$

474. The student should note and compare the two series (4) and (7) of the last article; viz.,

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \ldots + \frac{x^r}{\lfloor r} + \ldots \text{ for } \textit{all} \text{ finite values of } x,$$

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r-1}\frac{x^r}{r} + \ldots \text{ for } x \text{ not} > 1.$$

The *power* of e is expressed by (1) *factorial* denominators and (2) *positive* signs.

The *logarithm* to base e is expressed by (1) *simple numeral* denominators and (2) *alternate* signs.

It is important to note that the series on the right is to be equated to the *one positive value* of e^x in the first case; and to the *one positive or negative value* of $\log_e(1+x)$ in the second case.

475. The series for $\log_e(1+x)$ gives directly the logarithms only of (positive) numbers up to 2 inclusive.

Now any number greater than 1 is the reciprocal of some number less than 1: and $\log n = -\log\frac{1}{n}$. Hence the series gives almost directly the logarithms of any number whatever.

Thus, if $p > q$,

$$\log_e \frac{p}{q} = -\log \frac{q}{p} = -\log\left(1 - \frac{p-q}{p}\right)$$

$$= \frac{p-q}{p} + \tfrac{1}{2}\cdot\left(\frac{p-q}{p}\right)^2 + \tfrac{1}{3}\left(\frac{p-q}{p}\right)^3 + \tfrac{1}{4}\left(\frac{p-q}{p}\right)^4 + \ldots$$

Example. If $x > 1$, show that

$$\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \ldots\ldots = \frac{1}{1+x} + \frac{1}{2(1+x)^2} + \frac{1}{3(1+x)^3} + \ldots\ldots$$

476. The series for $\log_e(1+x)$ does not converge rapidly unless x is very small. Hence it is not immediately useful for the arithmetic calculation of logarithms.

But we have

$$\log_e(1+x) = +x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \ldots$$

$$\log_e(1-x) = -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \ldots$$

$\therefore$ by subtraction, (if x is not > 1 numerically)

$$\log_e \frac{1+x}{1-x} = 2\left(x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \ldots\right)\ldots\ldots\ldots\ldots\ldots(8).$$

If now m and n are positive and $m > n$, then $\dfrac{m-n}{m+n}$ is positive and < 1. Put $x = \dfrac{m-n}{m+n}$ in (8).

Thus $\log_e \dfrac{m}{n} = 2\left\{\dfrac{m-n}{m+n} + \tfrac{1}{3}\left(\dfrac{m-n}{m+n}\right)^3 + \tfrac{1}{5}\left(\dfrac{m-n}{m+n}\right)^5 + \ldots\right\}\ldots(9).$

Hence we can find in succession, from $\log_e 1 = 0$, the logarithms of any other numbers by a rapidly converging series.

Thus put $n = 1$ and $m = 2$. Thus

$$\log_e 2 = 2\left\{\tfrac{1}{3} + \tfrac{1}{3}\left(\tfrac{1}{3}\right)^3 + \tfrac{1}{5}\left(\tfrac{1}{3}\right)^5 + \ldots\right\}.$$

Put $n = 2$ and $m = 3$. Thus

$$\log_e 3 - \log_e 2 = 2\left\{\tfrac{1}{5} + \tfrac{1}{3}\left(\tfrac{1}{5}\right)^3 + \tfrac{1}{5}\left(\tfrac{1}{5}\right)^5 + \ldots\right\}$$

and so on.

477. In order to calculate logarithms to base 10, we have to note that

$$\log_{10} n = \log_e n \div \log_e 10.$$

Hence every log to base e must be multiplied by the constant modulus $1 \div \log_e 10$ to give the corresponding log to base 10.

As an example in the calculation of logarithms, we will find the value of $1 \div \log_e 10$ to 8 decimal places.

478. *To find the modulus* $1 \div \log_e 10$.

We have

$$\log_e 10 = \log_e \tfrac{5}{4} + \log_e 8 = \log_e \tfrac{5}{4} + 3 \log_e \tfrac{2}{1}.$$

In Art. 476, (9), put $m = 5$ and $n = 4$; also put $m = 2$ and $n = 1$. Thus

$$\log_e \tfrac{5}{4} = 2\{\tfrac{1}{9} + \tfrac{1}{3}(\tfrac{1}{9})^3 + \tfrac{1}{5}(\tfrac{1}{9})^5 + \tfrac{1}{7}(\tfrac{1}{9})^7 + \ldots\},$$
$$3 \log_e 2 = 2\{1 + \tfrac{1}{3}(\tfrac{1}{3})^2 + \tfrac{1}{5}(\tfrac{1}{3})^4 + \tfrac{1}{7}(\tfrac{1}{3})^6 + \ldots\},$$
$$= 2\{1 + \tfrac{1}{3} \cdot \tfrac{1}{9} + \tfrac{1}{5}(\tfrac{1}{9})^2 + \tfrac{1}{7}(\tfrac{1}{9})^3 + \ldots\}.$$

Thus we have expressed $\log_e 10$ in powers of $\frac{1}{9}$.

By successively dividing by 9 we may place in one column the several powers of $\frac{1}{9}$, and dividing these by the required numbers, we may place the results in a second column and add. Thus

$1 \div 9 = \cdot 111111111$	$\div 1 = \cdot 111111111$
	$\div 3 = \cdot 037037037$
$1 \div 9^2 = \cdot 012345679$	$\div 5 = \cdot 002469136$
$1 \div 9^3 = \cdot 001371742$	$\div 3 = \cdot 000457247$
	$\div 7 = \cdot 000195963$
$1 \div 9^4 = \cdot 000152416$	$\div 9 = \cdot 000016935$
$1 \div 9^5 = \cdot 000016935$	$\div 5 = \cdot 000003387$
	$\div 11 = \cdot 000001540$
$1 \div 9^6 = \cdot 000001882$	$\div 13 = \cdot 000000145$
$1 \div 9^7 = \cdot 000000209$	$\div 7 = \cdot 000000030$
	$\div 15 = \cdot 000000014$

$\therefore$ (sum adding 1) $= 1 \cdot 151292545$

This result has to be multiplied by 2.

Thus $\log_e 10 = 2{\cdot}30258509.$

And, by division,

$$\text{modulus} = \frac{1}{\log_e 10} = \log_{10} e = {\cdot}43429448.$$

479. Logarithms to the base 10 are called *Common* Logarithms: those to the base e are called *Napierian* Logarithms.

480. *To prove that* e *is incommensurable.*

If possible, let $e = \frac{m}{n}$, where m and n are *finite* integers. Thus

$$\frac{m}{n} = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \dots$$

Multiply both sides by $\lfloor n$; then

$m \lfloor n-1 = $ an integer

$$+ \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \dots$$

But the series

$$\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \dots$$

is less than the geometrical series

$$\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \dots$$

$$\text{i.e.} < \frac{1}{n+1} \cdot \frac{1}{1 - 1/(n+1)} \text{ i.e.} < \frac{1}{n+1-1} \text{ i.e.} < \frac{1}{n}.$$

This series is therefore a fraction less than 1.

Thus the difference of two integers is less than 1 but greater than 0: which is impossible.

Hence e cannot be expressed as a fraction with finite integral numerator and denominator; i.e. it is irrational or incommensurable (with unity).

Examples XVIII.

[Here 'the series $\phi(n)$' is used as an abbreviation for 'the series whose n^{th} term is $\phi(n)$'.]

1. The series $n^y x^n$, where y is finite and x positive, is convergent if $x < 1$, and divergent if $x > 1$.

2. Prove the theorem of Art. 454, in the converse form; viz. "The series $\phi(n)$ is convergent or divergent according as the series $b^n \phi(b^n)$ is convergent or divergent, where $\phi(n)$ is positive and decreases as n increases, and $b > 1$."*

3. Hence show that the series $\log_n x$, where x is positive, is divergent.

4. From the known theorem that the series x^n is convergent or divergent according as x is or is not less than 1, show that the series u_n is convergent or divergent according as *the limit*, when n is infinite, of $\sqrt[n]{u_n}$ is or is not less than 1.

5. Hence the series $(a_0 + a_1/n + a_2/n^2 + \ldots)^n$ is convergent or divergent according as a_0 is or is not less than 1.

6. From the known theorem that the series n^x is convergent or divergent according as x is or is not less than -1, show that the series u_n is convergent or divergent according as *the limit*, when n is infinite, of $\log_n u_n$ is or is not less than -1.

7. Hence the series $n^{a_0 + a_1/n + a_2/n^2 + \cdots}$ is convergent or divergent according as a_0 is or is not less than -1.

8. The series

$$\frac{1}{4} + \frac{1.3}{4.6} + \frac{1.3.5}{4.6.8} + \frac{1.3.5.7}{4.6.8.10} + \ldots \text{ is convergent and} = 1.$$

9. The series

$$\frac{2}{6} + \frac{2.5}{6.9} + \frac{2.5.8}{6.9.12} + \frac{2.5.8.11}{6.9.12.15} + \ldots \text{ is convergent and} = 2.$$

* This is generally the form in which the theorem should be used, when the series, whose convergency has to be determined, is *given*.

10. The series $x \sin\theta + x^2 \sin 2\theta + \ldots + x^n \sin n\theta + \ldots$

A. If $x > 1$, is divergent.

B. If $x < 1$, it is convergent and $= \dfrac{x \sin\theta}{1 - 2x\cos\theta + x^2}$.

C. If $x = 1$, and θ/π is incommensurable, it is finite but wholly indeterminate.

D. If $x = 1$, and $\theta/\pi = p/q$ where p and q are prime to one another, it $= \frac{1}{2}\cot\frac{1}{2}\theta - \frac{1}{2}\operatorname{cosec}\frac{1}{2}\theta \cos(n+\frac{1}{2})\theta$, an expression which has $\frac{1}{2}(q+1)$ values if p is even, but q values if p is odd.

11. The series $1 + x\cos\theta + x^2\cos 2\theta + \ldots + x^n \cos n\theta + \ldots$

A. If $x > 1$, is divergent.

B. If $x < 1$, it is convergent and $= \dfrac{1 - x\cos\theta}{1 - 2x\cos\theta + x^2}$.

C. If $x = 1$, and θ/π is incommensurable, it is finite but wholly indeterminate.

D. If $x = 1$, and $\theta/\pi = p/q$ where p and q are prime to one another, it $= \frac{1}{2}\operatorname{cosec}\frac{1}{2}\theta \sin(n+\frac{1}{2})\theta + \frac{1}{2}$, an expression which has $q+1$ values if $q+p$ is even, but q values if $q+p$ is odd.

12. The series $\dfrac{n^n x^n}{\lfloor n}$ is convergent or divergent according as x is or is not less than e^{-1}.

13. $$\sec^n\theta = 1 + n\frac{\sin^2\theta}{2} + \frac{n(n+2)}{\lfloor 2}\left(\frac{\sin^2\theta}{2}\right)^2 + \frac{n(n+2)(n+4)}{\lfloor 3}\left(\frac{\sin^2\theta}{2}\right)^3 + \ldots$$

14. $$\sec^n\theta = 1 + \frac{n}{2}\tan^2\theta + \frac{n(n-2)}{2.4}\tan^4\theta + \frac{n(n-2)(n-4)}{2.4.6}\tan^6\theta + \ldots$$

or $$= \tan^n\theta + \frac{n}{2}\tan^{n-2}\theta + \frac{n(n-2)}{2.4}\tan^{n-4}\theta + \ldots$$

according as θ is less or greater than $\frac{1}{4}\pi$.

15. Prove directly that the series

$$1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\dots$$

is convergent for all finite values of x.

16. Prove directly that the series

$$x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\dots$$

is convergent if x has any value from -1 to $+1$, excluding the first but including the last.

Hence show that $\log_e 2=\dfrac{1}{1.2}+\dfrac{1}{3.4}+\dfrac{1}{5.6}+\dots$

17. $\dfrac{e}{2}=1+\dfrac{2}{\lfloor 3}+\dfrac{3}{\lfloor 5}+\dfrac{4}{\lfloor 7}+\dfrac{5}{\lfloor 9}+\dots$

18. $\dfrac{2}{\lfloor 3}+\dfrac{4}{\lfloor 5}+\dfrac{6}{\lfloor 7}+\dots=\dfrac{1}{e}$.

19. $\log_e 2=\frac{2}{3}\left(1+\dfrac{1}{3.9}+\dfrac{1}{5.9^2}+\dfrac{1}{7.9^3}+\dots\right)$.

20. $\log_e 3=1+\dfrac{1}{3.9}+\dfrac{1}{5.9^2}+\dfrac{1}{7.9^3}+\dots$

$$+\frac{1}{17}+\frac{1}{3.17^3}+\frac{1}{5.17^5}+\dots$$

21. $\log_e 7=\dfrac{2}{9}+\dfrac{2}{3.9^3}+\dfrac{2}{5.9^4}+\dots$

$$+\frac{5}{3}+\frac{5}{3.3^3}+\frac{5}{5.3^5}+\frac{5}{7.3^7}+\dots$$

$$-\frac{1}{99}-\frac{1}{3.99^3}-\frac{1}{5.99^5}-\dots$$

22. $\log_e \sqrt{10}=\Sigma_1^\infty\left[\dfrac{1}{2n+1}+\dfrac{1}{2n}\{1-(-1)^n\}\right]\dfrac{1}{9^n}$.

23. $\log_e(x+1)-\log_e x$

$$=2\left\{\frac{1}{2x+1}+\frac{1}{3(2x+1)^3}+\frac{1}{5(2x+1)^5}+\dots\right\}.$$

24. $\log_e(x+1) - \log_e(x-1) = 2\left(\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \ldots\right)$.

25. $2\log_e x - \log_e(x+1) - \log_e(x-1)$
$= 2\left\{\frac{1}{2x^2-1} + \frac{1}{3(2x^2-1)^3} + \frac{1}{5(2x^2-1)^5} + \ldots\right\}$.

26. $\log_e n = m\{(\sqrt[m]{n}-1) - \frac{1}{2}(\sqrt[m]{n}-1)^2 + \frac{1}{3}(\sqrt[m]{n}-1)^3 - \ldots\}$.

27. If $\{\log_e(1+x)\}^2 = x^2 - \frac{2}{3}a_3x^3 + \frac{2}{4}a_4x^4 - \frac{2}{5}a_5x^5 + \ldots$

then $$a_{r+1} = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots + \frac{1}{r}.$$

28. If α, β be the roots of $px^2 + qx + r = 0$,
$\log_e(p - qx + rx^2) = \log_e p + (\alpha+\beta)x - \frac{1}{2}(\alpha^2+\beta^2)x^2 + \frac{1}{3}(\alpha^3+\beta^3)x^3 - \ldots$

29. From the known expansion
$$\tfrac{1}{2}\cdot\frac{1-x^2}{1-2x\cos\theta+x^2} = \tfrac{1}{2} + x\cos\theta + x^2\cos 2\theta + x^3\cos 3\theta + \ldots$$
show that
$$-\tfrac{1}{2}\log_e(1-2x\cos\theta+x^2) = x\cos\theta + \tfrac{1}{2}x^2\cos 2\theta + \tfrac{1}{3}x^3\cos 3\theta + \ldots$$

30. If n is odd, and $\phi = \pi/n$, then
$\log_e(1-2x\cos 2\phi + x^2) + \log_e(1-2x\cos 4\phi + x^2)$
$+ \ldots + \log_e\{1-2x\cos(n-1)\phi + x^2\}$
$= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 \ldots - (x^n + \frac{1}{2}x^{2n} + \frac{1}{3}x^{3n} + \ldots)$.

31. $$\frac{1}{\sec 2\phi - \tan 2\phi\cos\theta}$$
$$= 1 + 2\tan\phi\cos\theta + 2\tan^2\phi\cos 2\theta + 2\tan^3\phi\cos 3\theta + \ldots$$
if ϕ lies between 0 and $\frac{1}{4}\pi$.

32. Expand $\frac{1}{1+y\cos\theta}$ in a series of cosines of multiples of θ: and show that the constant term is $\frac{1}{\sqrt{(1-y^2)}}$, and the co-efficient of $\cos n\theta$ is $\frac{2\lambda^n}{\sqrt{(1-y^2)}}$, where λ is the larger root of $x^2 + 2x/y + 1 = 0$.

CHAPTER XIX.

RELATIONS BETWEEN THE CIRCULAR MEASURE AND THE TRIGONOMETRICAL RATIOS OF ANGLES.

§ 1. Relations of Inequality.

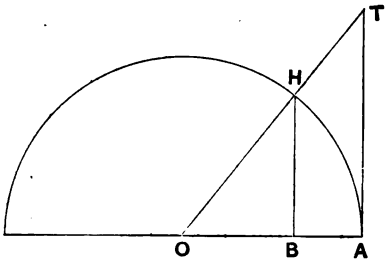
481. The symbol θ^c represents an angle containing θ radians; i.e. an angle whose circular measure is θ, the unit of circular measurement being the radian. When we are referring to the angle θ^c it is often convenient to drop the symbol for the unit and to write θ simply to denote the angle. The context will always show when θ means "the angle whose circular measure is θ," i.e. "the angle equal to θ radians." Without such context, θ denotes simply a *number*. See Chapter II.

482. If θ represents the circular measure of an angle, we know, by Art. 51, that $\theta =$ the ratio of the arc—subtended by the angle at the centre of a circle—to the radius.

We shall now apply the proposition of Art. 59 to a comparison between the circular measure and the trigonometrical ratios of an angle.

483. *The circular measure of an acute angle lies in magnitude between the* sine *and the* tangent *of the angle.*

At the centre O of a circle, let the acute angle AOH be equal to θ radians.



Draw HB, AT at right-angles to OA.

Then, by Art. 59,

$$\text{perp. } BH < \text{arc } AH < \text{tangent } AT.$$

Divide each by the radius of the circle. Thus

$$\frac{BH}{OH} < \frac{AH}{OA} < \frac{AT}{OA},$$

$$\text{i.e. } \sin\theta < \theta < \tan\theta \quad \ldots\ldots(1).$$

Also we have $$\cos\theta < 1 \quad \ldots\ldots(2).$$

These are the first approximations to the values of the ratios of an acute angle in terms of its circular measure.

The student must particularly observe that in the above inequalities θ, *standing by itself*, means a *number*; but θ, *standing after sin, cos*, or *tan*, means an *angle*. The inequalities are, of course, relations between mere numbers or ratios.

484. By expressing the ratios of θ in terms of those of $\frac{1}{2}\theta$, we obtain closer relations between the ratios and the circular measure. See next three articles.

We confine ourselves in this section (except in Arts. 490 and 493) to *acute angles*.

485. We have

$$\sin\theta = 2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta = 2\tan\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\theta = 2\tan\tfrac{1}{2}\theta\,(1-\sin^2\tfrac{1}{2}\theta).$$

Now, by (1) Art. 483,

$$\tan\tfrac{1}{2}\theta > \tfrac{1}{2}\theta \text{ and } \sin\tfrac{1}{2}\theta < \tfrac{1}{2}\theta,$$

$\therefore$ substituting in the above,

$$\textit{à fortiori}, \sin\theta > 2 \cdot \frac{\theta}{2}\left(1 - \frac{\theta^2}{4}\right), \text{ i.e. } > \theta - \frac{\theta^3}{4} \ldots\ldots (3).$$

Hence $\sin\theta$ *lies between* $\theta - \frac{1}{4}\theta^3$ *and* θ.

486. We have $\cos\theta = 1 - 2\sin^2\frac{1}{2}\theta$,

$\therefore$ substituting from (1), $\cos\theta > 1 - 2(\frac{1}{2}\theta)^2$, i.e. $> 1 - \frac{\theta^2}{2} \ldots\ldots (4)$;

but substituting from (3), $\cos\theta < 1 - 2\left(\frac{\theta}{2} - \frac{\theta^3}{32}\right)^2$,

$$\text{i.e. } < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{16} - \frac{\theta^6}{512},$$

$$\textit{à fortiori} < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{16} \ldots\ldots\ldots\ldots (5).$$

Hence $\cos\theta$ *lies between* $1 - \frac{\theta^2}{2}$ *and* $1 - \frac{\theta^2}{2} + \frac{\theta^4}{16}$.

487. We have, by division,

$$\tan\theta = \frac{2\tan\frac{1}{2}\theta}{1 - \tan^2\frac{1}{2}\theta} = 2\tan\tfrac{1}{2}\theta\,(1 + \tan^2\tfrac{1}{2}\theta + \tan^4\tfrac{1}{2}\theta + \ldots),$$

a convergent series, since $\frac{1}{2}\theta^c < 45°$, and $\therefore$ $\tan\frac{1}{2}\theta < 1$.

$\therefore$ substituting from (1), $\tan\theta > \theta + \frac{\theta^3}{4} + \frac{\theta^5}{16} + \ldots + \frac{\theta^{2n+1}}{4^n} + \ldots$

$\therefore$ *à fortiori*, $\tan\theta >$ sum of any number of these terms.

Limits.

488. *The limit of* $\frac{\sin\theta}{\theta}$ *and of* $\frac{\tan\theta}{\theta}$, *when* θ *is indefinitely diminished, is unity.*

For $\sin\theta$, θ, $\tan\theta$ are in ascending order of magnitude.

Hence, dividing each by $\sin\theta$ and by $\tan\theta$, both

$$1, \frac{\theta}{\sin\theta}, \sec\theta; \text{ and } \cos\theta, \frac{\theta}{\tan\theta}, 1,$$

are in ascending order of magnitude.

But, as θ decreases, $\sec\theta$ and $\cos\theta$ may each be made as near to unity as we please. Hence

$\theta/\sin\theta$ which lies between 1 and $\sec\theta$;

and $\theta/\tan\theta$ which lies between $\cos\theta$ and 1;

may each be made as near to unity as we please.

Thus the limits, when θ is indefinitely diminished, of $\frac{\theta}{\sin\theta}$, $\frac{\sin\theta}{\theta}$, $\frac{\theta}{\tan\theta}$, $\frac{\tan\theta}{\theta}$ are each unity.

489. *The limit of the ratio of the sines or tangents of two vanishing angles is equal to the limit of the ratio of the two angles.*

For let $\frac{\sin\theta}{\theta} = 1 - x$ and $\frac{\sin\phi}{\phi} = 1 - y$.

Then $\frac{\sin\theta}{\sin\phi} = \frac{\theta(1-x)}{\phi(1-y)}$.

Now, as θ and ϕ are indefinitely diminished, x and y are also indefinitely diminished by last article.

Hence, limit of ratio $\frac{\sin\theta}{\sin\phi}$ = limit of ratio $\frac{\theta}{\phi}$.

Similarly limit of $\frac{\tan\theta}{\tan\phi}$ = limit of $\frac{\theta}{\phi}$.

490. *The limit of the ratio of* $\sin k\theta$ *to* k, *when* k *is indefinitely diminished and* θ *is finite, is* θ.

For $\frac{\sin k\theta}{k} = \theta \cdot \frac{\sin k\theta}{k\theta}$.

Now, since θ is finite, $\therefore$ $k\theta$ is indefinitely diminished when k is indefinitely diminished.

$\therefore$ limit of $\sin k\theta \div k\theta$ is unity.

$\therefore$ limit of $\sin k\theta \div k$ is θ.

491. The results of the last three articles should be carefully examined.

In Art. 488, observe *first*, that θ, *not* $\sin\theta$, is to be made indefinitely small. If $\theta = \pi$ for instance, $\sin\theta$ vanishes, but the limit of $\sin\theta \div \theta$ would be, *not unity*, but *zero*.

Observe *secondly*, that θ is the *circular measure* of the angle not any other measure. For instance, if the angle θ radians $= A$ degrees, then

$$A = \frac{180}{\pi} \times \theta.$$

$$\therefore \frac{\sin A^0}{A} = \frac{\pi \sin \theta^c}{\theta \times 180}, \text{ whose limit is } \frac{\pi}{180}, \textit{ not unity.}$$

In Art. 489, however,

$$\text{since } \frac{\theta}{\phi} = \frac{\theta \text{ radians}}{\phi \text{ radians}} = \frac{A \text{ degrees}}{B \text{ degrees}} \text{ (say)},$$

$$\therefore \text{limit of } \frac{\sin A^0}{\sin B^0} = \text{limit of } \frac{A}{B}.$$

As before in Art. 488, A and B here must diminish indefinitely, not only $\sin A^0$ and $\sin B^0$.

In Art. 490, on the other hand, θ may have any finite value. Thus θ might be the circular measure of an angle greater than two right-angles. For here the angle which is to be made small is $k\theta$, not θ; and provided θ is finite, $k\theta$ is made small by diminishing k.

492. The results of the three articles, 488, 489, 490 are usefully summarised in the statement

$$\textit{When } \theta = 0, \textit{ sin } \theta = \theta.$$

But in this statement we must distinguish two distinct propositions; viz.,

(1) When θ is **small**, $\sin\theta$ is **approximately** equal to θ.

(2) When θ is **indefinitely diminished**, the limit of $\frac{\sin\theta}{\theta}$ is **exactly** unity.

The first proposition may be used in arithmetical and approximate calculations: the second in algebraical and exact theorems.

The propositions in the following articles will exemplify these distinctions.

493. *To show that, when n is indefinitely increased and θ is finite, the limit of*

$$\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{4}\cdot\cos\frac{\theta}{8}\dots\cos\frac{\theta}{2^n} \text{ is } \frac{\sin\theta}{\theta}.$$

For $$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2^2\sin\frac{\theta}{2^2}\cos\frac{\theta}{2}\cos\frac{\theta}{2^2},$$

and so on.

Thus $$\sin\theta = 2^n\sin\frac{\theta}{2^n}\left(\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\dots\cos\frac{\theta}{2^n}\right).$$

But $$2^n\sin\frac{\theta}{2^n} = \theta\cdot\frac{\sin\theta/2^n}{\theta/2^n}.$$

And, as in Art. 490, when n is indefinitely increased and $\therefore$ $\theta/2^n$ indefinitely decreased, the limit of the above is θ.

Hence, in the limit, $$\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\cdot\cos\frac{\theta}{2^3}\dots = \frac{\sin\theta}{\theta}.$$

Example. Put $\theta = \frac{1}{2}\pi$. Then $\sin\theta = 1$; $2\cos\frac{1}{2}\theta = \sqrt{2}$; $2\cos\frac{1}{4}\theta = \sqrt{(2+\sqrt{2})}$; $2\cos\frac{1}{8}\theta = \sqrt{\{2+\sqrt{(2+\sqrt{2})}\}}$; and so on.

Thus, the product of the endless series of factors

$$\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{(2+\sqrt{2})}}{2}\cdot\frac{\sqrt{\{2+\sqrt{(2+\sqrt{2})}\}}}{2}\dots\dots = \frac{2}{\pi}.$$

494. *To show that* $\sin\theta > \theta - \frac{1}{6}\theta^3$, *if* θ *denotes an acute angle.*

By Art. 486, $\cos\theta > 1 - \frac{1}{2}\theta^2$.

$$\therefore \cos\frac{\theta}{2}\cos\frac{\theta}{4} > \left(1-\frac{\theta^2}{2^3}\right)\left(1-\frac{\theta^2}{2^5}\right),$$

$$\text{à fortiori} > 1 - \frac{\theta^2}{2^3} - \frac{\theta^2}{2^5}.$$

Proceeding in this way,

$$\cos\frac{\theta}{2}\cos\frac{\theta}{4}\dots\cos\frac{\theta}{2^n} > 1 - \frac{\theta^2}{2}\left(\frac{1}{4}+\frac{1}{4^2}+\dots+\frac{1}{4^n}\right)$$

$$> 1 - \frac{\theta^2}{8}\cdot\frac{1-1/4^n}{1-1/4}.$$

$$> 1 - \frac{\theta^2}{6} + \frac{\theta^2}{6 \cdot 4^n},$$

$$\text{à fortiori} > 1 - \frac{\theta^2}{6}.$$

Now as in the last article,

$$\frac{\sin\theta}{\theta} = \frac{\sin\theta/2^n}{\theta/2^n} \cdot \cos\frac{\theta}{2}\cos\frac{\theta}{4}\ldots\cos\frac{\theta}{2^n}.$$

Thus, $\frac{\sin\theta}{\theta}$ is *less* than the product of these cosines, but may be made as nearly equal to it as we please by increasing n.

Hence $$\frac{\sin\theta}{\theta} > 1 - \frac{\theta^2}{6},$$

$$\text{i.e. } \sin\theta > \theta - \tfrac{1}{6}\theta^3.$$

495. *To show that* $\cos\theta < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$.

We have $$\cos\theta = 1 - 2\sin^2\tfrac{1}{2}\theta$$

$$< 1 - 2\left(\frac{\theta}{2} - \frac{\theta^3}{48}\right)^2 \text{ by above,}$$

$$\text{à fortiori} < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}.$$

496. We have $\tan\theta = \dfrac{2\tan\frac{1}{2}\theta}{1 - \tan^2\frac{1}{2}\theta}$,

$$\therefore \tan\theta - 2\tan\tfrac{1}{2}\theta = \frac{2\tan^3\frac{1}{2}\theta}{1 - \tan^2\frac{1}{2}\theta} > \frac{2(\frac{1}{2}\theta)^3}{1 - (\frac{1}{2}\theta)^2} \text{ by Art. 483.}$$

$$> \frac{\theta \cdot \frac{1}{4}\theta^2}{1 - \frac{1}{4}\theta^2},$$

$$\therefore \frac{\tan\theta}{\theta} - \frac{\tan\frac{1}{2}\theta}{\frac{1}{2}\theta} > \frac{\theta^2}{4} + \left(\frac{\theta^2}{4}\right)^2 + \ldots + \left(\frac{\theta^2}{4}\right)^n + \ldots$$

$$\frac{\tan\frac{1}{2}\theta}{\frac{1}{2}\theta} - \frac{\tan\frac{1}{4}\theta}{\frac{1}{4}\theta} > \frac{1}{4}\cdot\frac{\theta^2}{4} + \frac{1}{4^2}\cdot\left(\frac{\theta^2}{4}\right)^2 + \ldots + \frac{1}{4^n}\cdot\left(\frac{\theta^2}{4}\right)^n + \ldots$$

$$\frac{\tan\frac{1}{4}\theta}{\frac{1}{4}\theta} - \frac{\tan\frac{1}{8}\theta}{\frac{1}{8}\theta} > \frac{1}{4^2}\cdot\frac{\theta^2}{4} + \frac{1}{4^4}\left(\frac{\theta^2}{4}\right)^2 + \ldots + \frac{1}{4^{2n}}\cdot\left(\frac{\theta^2}{4}\right)^n + \ldots$$

and so on.

Increasing indefinitely the number of these inequalities, adding them, and observing that the term on the right involving θ^{2n} is

$$\left(\frac{\theta^2}{4}\right)^n\left(1+\frac{1}{4^n}+\frac{1}{4^{2n}}+\ldots\right)=\frac{\theta^{2n}}{4^n}\cdot\frac{1}{1-1/4^n}=\frac{\theta^{2n}}{4^n-1},$$

and that the limit of $\dfrac{\tan\theta/2^r}{\theta/2^r}$ on the left is 1, we have

$$\frac{\tan\theta}{\theta}-1>\frac{\theta^2}{3}+\frac{\theta^4}{15}+\frac{\theta^6}{63}+\ldots+\frac{\theta^{2n}}{4^n-1}+\ldots$$

$$\therefore\ \tan\theta>\theta+\frac{\theta^3}{3}+\frac{\theta^5}{15}+\ldots+\frac{\theta^{2n+1}}{4^n-1}+\ldots$$

and *à fortiori* $\tan\theta >$ sum of any number of these terms.

497. The results in these last three articles are nearer approximations to the values of the ratios than the corresponding results in Arts. 485, 486, 487.

§ 2. Ratios in terms of Circular Measure.

498. *To find the limit of* $\left(\sec\dfrac{\theta}{n}\right)^n$ *when* n *is indefinitely increased.*

Since $\sec\dfrac{\theta}{n}=1$ when n is indefinitely large, we have here to evaluate a limit of the form 1^∞—the index being a function of the base. See Art. 468.

Now, if x is any quantity greater than 1, the following three quantities are in ascending order of magnitude, viz.,

$$1\ ;\quad 1+\tan^2\frac{\theta}{n}\ ;\quad \left(1-\frac{1}{x}\right)^{-x\tan^2\frac{\theta}{n}}.$$

For if the last quantity is expanded by the binomial theorem, its first two terms are $1+\tan^2\dfrac{\theta}{n}$ and the remaining terms are all *positive*.

Raising each to the power $\frac{1}{2}n$, the following three are in ascending order, viz.,

$$1\,; \quad \left(\sec\frac{\theta}{n}\right)^{n}; \quad \left(1-\frac{1}{x}\right)^{-\frac{nx}{2}\tan^2\frac{\theta}{n}}.$$

But limit of $n\tan\frac{\theta}{n} = \theta \,.\, \frac{\tan\theta/n}{\theta/n}$ is θ.

$\therefore$ limit of $-\frac{n}{2}\tan^2\frac{\theta}{n}$ is 0.

$\therefore \left(\sec\frac{\theta}{n}\right)^{n}$ lies in the limit between 1 and $\left(1-\frac{1}{x}\right)^{0}$;

i.e. in the limit $\left(\sec\frac{\theta}{n}\right)^{n}$ is 1.

$\therefore$ in the limit $\left(\cos\frac{\theta}{n}\right)^{n}$ is 1.

499. *To find the limit of* $\left(\frac{\sin\theta/n}{\theta/n}\right)^{n}$ *and* $\left(\frac{\tan\theta/n}{\theta/n}\right)^{n}$ *when* n *is indefinitely increased.*

Since $\left(\sin\frac{\theta}{n}\right)^{n}$, $\left(\frac{\theta}{n}\right)^{n}$, $\left(\tan\frac{\theta}{n}\right)^{n}$ are in ascending order of magnitude,

$\therefore$ dividing by $\left(\sin\frac{\theta}{n}\right)^{n}$; $\quad 1, \left(\frac{\theta/n}{\sin\theta/n}\right)^{n}, \left(\sec\frac{\theta}{n}\right)^{n}$

are in ascending order of magnitude,

and dividing by $\left(\tan\frac{\theta}{n}\right)^{n}$; $\left(\cos\frac{\theta}{n}\right)^{n}, \left(\frac{\theta/n}{\tan\theta/n}\right), 1$

are in ascending order of magnitude.

Hence, by the last article, the limits of $\left(\frac{\theta/n}{\sin\theta/n}\right)^{n}$ and of $\left(\frac{\theta/n}{\tan\theta/n}\right)^{n}$ and of their reciprocals are each unity.

500. The above limits—and others of the form 1^{∞}—may be found from the known theorem that the limit of $\left(1+\frac{1}{n}\right)^n$ is e.

Thus $(\sec\theta)^{x^m+\theta^m} = \left[(1+\tan^2\theta)^{\cot^2\theta}\right]^{\frac{1}{2}x^m\tan^2\theta+\theta^m}$.

Hence, when $\theta=0$, the expression in the square brackets becomes equal to e, and the limit of $(\sec\theta)^{x^m+\theta^m}$ is as follows:

If $m<2$, the limit $=e^0=1$.

If $m=2$, the limit $=e^{\frac{1}{2}x^2}$.

If $m>2$, the limit $=e^{\infty}=\infty$.

The particular case $\left(\cos\frac{\theta}{n}\right)^n$, when n is infinite, may be treated directly as follows:

$$\left(\cos\frac{\theta}{n}\right)^n = \left[\left(1-\sin^2\frac{\theta}{n}\right)^{-\operatorname{cosec}^2\frac{\theta}{n}}\right]^{-\frac{1}{2}n\sin^2\frac{\theta}{n}},$$

$=e^0=1$, when n is infinite.

For $n\sin\frac{\theta}{n}=\theta\,.\,\frac{\sin\theta/n}{\theta/n}=\theta$; $\therefore\ n\sin^2\frac{\theta}{n}=0$ in the limit.

501. *To expand* $\cos\theta$ *and* $\sin\theta$ *in powers of* θ. We have

$$\cos na = \cos^n a - \frac{n(n-1)}{\lfloor 2}\cos^{n-2}a\sin^2 a + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}\cos^{n-4}a\sin^4 a - \ldots$$

$$= \cos^n a\left[1-\frac{1-\frac{1}{n}}{\lfloor 2}\,.\,\frac{\tan^2 a}{\frac{1}{n^2}} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{\lfloor 4}\,.\,\frac{\tan^4 a}{\frac{1}{n^4}} - \ldots\right].$$

Now let $na=\theta$, so that $a=\frac{\theta}{n}$ and $\frac{1}{n}=\frac{a}{\theta}$.

Thus $\cos\theta = (\cos\theta/n)^n \times$

$$\left[1-\frac{1-\frac{1}{n}}{\lfloor 2}\,.\,\left(\frac{\tan a}{a}\right)^2\theta^2 + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{\lfloor 4}\,.\,\left(\frac{\tan a}{a}\right)^4.\theta^4 - \ldots\right].$$

Now let n increase indefinitely while θ remains constant and finite so that a decreases indefinitely.

Then, by Art. 498, limit of $\left(\cos\frac{\theta}{n}\right)^n$ is 1.

Also, since each of the factors

$$1-\frac{1}{n},\ 1-\frac{2}{n},\ldots\ 1-\frac{r}{n}$$

reaches its limit 1 independently of the number of them, therefore the limit of their product is 1.

Similarly since the diminution of a in $\left(\frac{\tan a}{a}\right)^r$ is independent of the index r, therefore the limit of $\left(\frac{\tan a}{a}\right)^r$ is 1. Thus

$$\cos\theta = 1-\frac{\theta^2}{\lfloor 2}+\frac{\theta^4}{\lfloor 4}-\frac{\theta^6}{\lfloor 6}+\ldots$$

Exactly in the same manner, we may prove that

$$\sin\theta=\theta-\frac{\theta^3}{\lfloor 3}+\frac{\theta^5}{\lfloor 5}-\frac{\theta^7}{\lfloor 7}+\ldots$$

502. The student should observe that the expansions of $\cos\theta$ and $\sin\theta$ in powers of θ are obtained from those of $\cos n\theta$ and $\sin n\theta$ in powers of $\cos\theta$ and $\sin\theta$ in the same way that the expansion of e^x is obtained from the binomial theorem. And hence that

The terms of cos θ and of sin θ are taken alternately but with alternate change of sign from the expansion of e^θ.

503. The above expansions of $\sin\theta$ and $\cos\theta$ hold for all finite values of θ whatever. For the ratio of any term to the preceding is $-\frac{\theta^2}{n(n-1)}$: which, when n is large enough, is clearly less than a quantity less than 1.

Now if we increase θ by 2π, the values of $\cos\theta$ and of $\sin\theta$ are unaltered; hence *the values of these series are unaltered by adding* 2π to θ.

In other words, the series

$$1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \ldots \quad \textit{and}\ \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \ldots$$

are *periodic.*

504. All the theorems with respect to the sine and cosine that have been investigated from the definition by means of an angle, may be proved from the above series. A few cases may be worked out.

505. Since every index in $\sin\theta$ is *odd*, $\therefore \sin(-\theta) = -\sin\theta$. Since every index in $\cos\theta$ is *even*, $\therefore \cos(-\theta) = \cos\theta$.

506. If θ measures an *acute* angle, $\theta^2 < \frac{1}{4}\pi^2$; $\therefore$ *à fortiori* $\theta^2 < 6$.

Now
$$\sin\theta = \theta - \left(\frac{\theta^3}{\lfloor 3} - \frac{\theta^5}{\lfloor 5}\right) - \left(\frac{\theta^7}{\lfloor 7} - \frac{\theta^9}{\lfloor 9}\right) - \ldots$$

$$= \left(\theta - \frac{\theta^3}{\lfloor 3}\right) + \left(\frac{\theta^5}{\lfloor 5} - \frac{\theta^7}{\lfloor 7}\right) + \ldots$$

Hence each of the above brackets is positive;

$\therefore \sin\theta < \theta$; but $> \theta - \frac{1}{6}\theta^3$; but $< \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5$; and so on.

Also
$$\theta - \sin\theta = \tfrac{1}{6}\theta^3 - \left(\frac{\theta^5}{\lfloor 5} - \frac{\theta^7}{\lfloor 7}\right) - \ldots < \tfrac{1}{6}\theta^3.$$

And
$$\sin\theta - (\theta - \tfrac{1}{6}\theta^3) = \tfrac{1}{120}\theta^5 - \left(\frac{\theta^7}{\lfloor 7} - \frac{\theta^9}{\lfloor 9}\right) - \ldots < \tfrac{1}{120}\theta^5.$$

Hence, since the series beginning at any term has the same sign as that term, *the difference between* $\sin\theta$ *and any number of its first terms is less than the first term omitted.*

Similar propositions hold with respect to the *cosine.*

507. The expansional values of functions investigated in this and the preceding chapter enable us to evaluate limits of the form $\frac{0}{0}$. Thus

Example 1. $\dfrac{x-\log_e(1+x)}{1-3x-(1-x)^3}$ when $x=0$.

The given fraction $= \dfrac{x-(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\ldots)}{1-3x-(1-3x+3x^2-x^3)}$

$$= \frac{\frac{1}{2}x^2-\frac{1}{3}x^3+\ldots}{-3x^2+x^3} = \frac{\frac{1}{2}-\frac{1}{3}x+\cdots}{-3+x} = -\tfrac{1}{6} \text{ when } x=0.$$

Example 2. $\dfrac{6\theta-6\sin\theta-\sin^3\theta}{\theta(3-4\cos\theta+\cos 2\theta)}$ when $\theta=0$.

The given fraction $= \dfrac{6\theta-6(\theta-\frac{1}{6}\theta^3+\frac{1}{120}\theta^5-\ldots)-(\theta-\frac{1}{6}\theta^3+\ldots)^3}{3\theta-4\theta(1-\frac{1}{2}\theta^2+\frac{1}{24}\theta^4-\ldots)+\theta(1-2\theta^2+\frac{2}{3}\theta^4-\ldots)}$

$$= \frac{\frac{1}{2}\theta^5-\frac{1}{20}\theta^5+\ldots}{\frac{2}{3}\theta^5-\frac{1}{6}\theta^5+\ldots} = \frac{\frac{9}{20}\theta^5+\cdots}{\frac{1}{2}\theta^5+\ldots} = \frac{\frac{9}{20}+\cdots}{\frac{1}{2}+\ldots} = \tfrac{9}{10} \text{ when } \theta=0.$$

The hyperbolic functions.

508. We have

$$\tfrac{1}{2}(e^\theta+e^{-\theta}) = 1+\frac{\theta^2}{\lfloor 2}+\frac{\theta^4}{\lfloor 4}+\frac{\theta^6}{\lfloor 6}+\ldots$$

$$\tfrac{1}{2}(e^\theta-e^{-\theta}) = \theta+\frac{\theta^3}{\lfloor 3}+\frac{\theta^5}{\lfloor 5}+\frac{\theta^7}{\lfloor 7}+\ldots$$

The above expansions of $\frac{1}{2}(e^\theta+e^{-\theta})$ and $\frac{1}{2}(e^\theta-e^{-\theta})$ are called the **Hyperbolic cosine** and the **Hyperbolic sine** respectively of θ; and are written shortly cosh θ and sinh θ.

Thus cos θ and sin θ are obtained from cosh θ and sinh θ respectively by changing the signs of the alternate coefficients of the several powers of θ.

Or cos θ and $\dfrac{\sin\theta}{\theta}$ are obtained from cosh θ and $\dfrac{\sinh\theta}{\theta}$ respectively by changing θ^2 into $-\theta^2$.

Further sinh $\theta \div$ cosh θ is called tanh θ: and so on.

509. The following table of comparison between the Trigonometrical (or Circular) and the Hyperbolic functions should be verified and studied.

It is assumed that x has any positive or negative value.

Circular Functions.	*Hyperbolic Functions.*
$\cos(-x) = +\cos x.$	$\cosh(-x) = +\cosh x.$
$\sin(-x) = -\sin x.$	$\sinh(-x) = -\sinh x.$
$\cos 0 = 1.$	$\cosh 0 = 1.$
$\sin 0 = 0.$	$\sinh 0 = 0.$
$\text{Limit } (x = 0)\ \dfrac{\sin x}{x} = 1.$	$\text{Limit } (x = 0)\ \dfrac{\sinh x}{x} = 1.$
$\text{Limit } \dfrac{\sin x}{\sin y} = \dfrac{x}{y}.$	$\text{Limit } \dfrac{\sinh x}{\sinh y} = \dfrac{x}{y}.$
$\cos x$ lies between -1 and $+1$.	$\cosh x$ lies between $+1$ and $+\infty$.
$\sin x$ lies between -1 and $+1$.	$\sinh x$ lies between $-\infty$ and $+\infty$.
$\cos x$ periodically decreases and increases: its values recurring at period 2π.	$\cosh x$ always increases numerically as x increases numerically.
$\sin x$ periodically decreases and increases: its values recurring at period 2π.	$\sinh x$ always increases algebraically as x increases algebraically.
$\cos^2 x + \sin^2 x = 1.$	$\cosh^2 x - \sinh^2 x = 1.$
$\cos(x+y) = \cos x \cos y - \sin x \sin y.$	$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y.$
$\sin(x+y) = \sin x \cos y + \cos x \sin y.$	$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y.$

These formulæ for the hyperbolic functions may be proved from their exponential or from their expansional equivalents.

Finally, in the limit, $\dfrac{\sinh x}{\sin y}$ (when x and y vanish) $= \dfrac{x}{y}$.

510. If $\cos x$ and $\sin x$ had been defined by their expansions in x, it could easily be shown that $\cos^2 x + \sin^2 x = 1$.

For, if each of the expansions is multiplied by itself and the two results added, $\pm$ the coefficient of $x^{2n} =$

$$\frac{1}{\lfloor 2n} - \frac{1}{1 \,.\, \lfloor 2n-1} + \frac{1}{\lfloor 2 \,.\, \lfloor 2n-2} - \frac{1}{\lfloor 3 \,\lfloor 2n-3} + \ldots = \frac{1}{\lfloor 2n}(1-1)^{2n} = 0,$$

by the binomial theorem for a positive integer.

Hence the only term remaining in $\cos^2 x + \sin^2 x$ is 1.

Similarly $\cosh^2 x - \sinh^2 x = 1$.

This latter result is, of course, obvious from the *exponential* equivalents of $\cosh x$ and $\sinh x$.

Geometrical Illustrations.

511. The student of Conic Sections should observe the following points; where we still define cos, sin, cosh, sinh by their expansions.

The equation to an ellipse, referred to its semi-axes a and b, is

$$x^2/a^2 + y^2/b^2 = 1.$$

Hence we may put $x = a\cos\phi$ and $y = b\sin\phi$, where ϕ has a variable value for the several points of the ellipse.

For, as shown in the last article, $\cos^2\phi + \sin^2\phi = 1$ for all values of ϕ.

The equation to a hyperbola, similarly referred, is

$$x^2/a^2 - y^2/b^2 = 1.$$

Hence we may put $x = a\cosh\phi$ and $y = b\sinh\phi$, where ϕ has a variable value for the several points of the hyperbola.

For, as shown in the last article, $\cosh^2\phi - \sinh^2\phi = 1$.

512. The student of the Integral Calculus should further observe that the area of the sector measured from 0 to any value ϕ is

$$\tfrac{1}{2}\int (x\,dy - y\,dx).$$

Now $d\sin\phi = \cos\phi\,d\phi$; $\quad d\cos\phi = -\sin\phi\,d\phi$,

$d\sinh\phi = \cosh\phi\,d\phi$; $\quad d\cosh\phi = \sinh\phi\,d\phi$,

as may be easily seen by differentiating the *expansions*.

Hence in the ellipse $\tfrac{1}{2}\int (x\,dy - y\,dx) = \tfrac{1}{2}ab\int (\cos^2\phi + \sin^2\phi)\,d\phi$

$$= \tfrac{1}{2}ab\int d\phi = \tfrac{1}{2}ab\phi.$$

In the hyperbola, $\tfrac{1}{2}\int (x\,dy - y\,dx) = \tfrac{1}{2}ab\int (\cosh^2\phi - \sinh^2\phi)\,d\phi$

$$= \tfrac{1}{2}ab\int d\phi = \tfrac{1}{2}ab\phi.$$

Hence, in either conic, ϕ denotes *the ratio of the sectorial area to the triangle whose base and altitude are respectively the semi-axes.*

Putting $a = b$, we have the circle or the rectangular hyperbola. In the case of the circle we have shown (Art. 54) that ϕ (the circular measure) = the ratio of the sector to the triangle whose base and altitude are each equal to the radius. And we now see that a similar result holds for any central conic.

513. *To express* $\tan\theta$ *and* $\cot\theta$ *as continued fractions in terms of* θ.

By the above expansions of $\cos\theta$ and $\sin\theta$, we have

$$\cos\theta = 1 - \frac{\theta^2}{2.1} + \frac{\theta^4}{2.4.1.3} - \frac{\theta^6}{2.4.6.1.3.5} + \dots$$

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{2.3} + \frac{\theta^4}{2.4.3.5} - \frac{\theta^6}{2.4.6.3.5.7} + \dots$$

Here we have arranged the denominators with the even factors before the odd.

Now let $f(n) =$

$$1 - \frac{\theta^2}{2(2n+1)} + \frac{\theta^4}{2.4(2n+1)(2n+3)} - \frac{\theta^6}{2.4.6(2n+1)(2n+3)(2n+5)} + \dots$$

Then $f(n-1) =$

$$1 - \frac{\theta^2}{2(2n-1)} + \frac{\theta^4}{2.4(2n-1)(2n+1)} - \frac{\theta^6}{2.4.6(2n-1)(2n+1)(2n+3)} + \dots$$

$\therefore f(n-1) - f(n) =$

$$\frac{-\theta^2}{(2n-1)(2n+1)}\left[1 - \frac{\theta^2}{2(2n+3)} + \frac{\theta^4}{2.4(2n+3)(2n+5)} - \dots\right] = \frac{-\theta^2 f(n+1)}{(2n-1)(2n+1)}.$$

$$\therefore \frac{(2n-1)f(n-1)}{f(n)} = (2n-1) - \frac{\theta^2}{\frac{(2n+1)f(n)}{f(n+1)}}.$$

Now $\cos\theta = f(0)$ and $\dfrac{\sin\theta}{\theta} = f(1)$.

$$\therefore \theta\cot\theta = \frac{1.f(0)}{f(1)} = 1 - \frac{\theta^2}{\frac{3f(1)}{f(2)}} = 1 - \frac{\theta^2}{3-}\,\frac{\theta^2}{5-}\,\frac{\theta^2}{7-}, \text{ \&c.}$$

and

$$\frac{\tan\theta}{\theta} = \frac{1}{\theta\cot\theta} = \frac{1}{1-}\,\frac{\theta^2}{3-}\,\frac{\theta^2}{5-}\,\frac{\theta^2}{7-}, \text{ \&c.}$$

§ 3. Resolution into Factors.

Endless Factorisation.

514. The *product* of an infinite number of *factors* may be used with the same limitations as the *sum* of an infinite number of *terms*. Thus

DEF. An endless product is said to be *convergent* if the product of its factors may be made as near as we please to *some one finite* quantity by increasing the number of the factors.

It is convenient to understand the word 'finite' here to exclude *zero* as well as infinity, so that if an endless product is convergent so also is its *reciprocal.*

Every factor may be assumed to be *positive*, since a change in sign of any one factor will not affect the magnitude of the product.

515. The properties of an endless product may always be investigated by taking its *logarithm*, by means of which it is reduced to an endless sum.

Thus the values 0 and ∞, which we exclude for convergency of a *product*, correspond respectively to $\log 0$ and $\log \infty$, i.e. to $-\infty$ and $+\infty$, which we exclude for convergency of a *sum.*

516. Corresponding to the distinction between positive and negative in terms, is the distinction between greater and less *than unity* in factors. Thus

Since the n^{th} factor of a product is equal to the ratio of the product of n factors to that of $n-1$ factors, *a product cannot be convergent unless, when* n *is infinite, the limit of the* n^{th} *term is unity.* (Cf. Art. 443.)

A few of the more important theorems may be given.

517. *A product whose factors are alternately greater and less than unity will be finite for all values of* n *if the product of each factor into the next is also alternately greater and less than unity.*

For consider the product $v_1v_2v_3 \ldots v_n \ldots$, where $v_1 > 1$; $v_2 < 1$; $\ldots$; and $v_1v_2 > 1$; $v_2v_3 < 1$; $\ldots$.

This product $= v_1v_2 \,.\, v_3v_4 \,.\, v_5v_6 \ldots$ and is $\therefore > 1$.

And it also $= v_1 \,.\, v_2v_3 \,.\, v_4v_5 \ldots$ and is $\therefore < v_1$. (Cf. Art. 446.)

518. *A product, all of whose factors are greater than unity, is* convergent *if the logarithm of every factor to the preceding as base is* less than some quantity less than unity.

Let $v_{n+1} < v_n^k$ where k is a positive quantity < 1.

Then raising the product $v_1v_2v_3 \ldots v_n \ldots$ to the power $1-k$, we have $v_1 . v_1^{-k} . v_2 . v_2^{-k} . v_3 . v_3^{-k} \ldots$ in which the factors are alternately greater and less than 1; and also the *pairs* of factors $v_1 . v_1^{-k}$; $v_1^{-k} . v_2$; $v_2^{-k} . v_3$, &c., are alternately greater and less than 1. (Cf. Art. 450.)

519. *If* v_n *always decreases as* n *increases and is always positive, then according as the* series *whose general* term *is* v_n *is convergent or divergent, so also is the* product *whose general* factor *is* $1 + v_n$.

For let $\log(1+v_n) = u_n$, so that $1 + v_n = e^{u_n}$.

Then $$\frac{e^{u_n}-1}{u_n} = 1 + \frac{u_n}{\underline{2}} + \frac{u_n^2}{\underline{3}} + \frac{u_n^3}{\underline{4}} + \ldots$$

This series cannot be zero or infinity for any finite value of u_n (including zero). That is, neither $\dfrac{v_n}{\log(1+v_n)}$ nor $\dfrac{\log(1+v_n)}{v_n}$ can be infinite for any value of n.

$\therefore$ (by Art. 453) according as the series whose general term is v_n is convergent or divergent, so is the series whose general term is $\log(1+v_n)$, and so also is the *product* whose general *factor* is $1 + v_n$.

The same argument applies to the product whose general factor is $1 - v_n$, where $v_n < 1$; which of course cannot be infinite but may be zero.

Example. The limit of the coefficient of x^n in the expansion of $(1+x)^m$ is zero or infinity according as m is greater or less than -1. For the coefficient of x^n is

$$= \left(1 - \frac{m+1}{1}\right)\left(1 - \frac{m+1}{2}\right)\left(1 - \frac{m+1}{3}\right)\ldots\left(1 - \frac{m+1}{n}\right)$$

This product is always divergent, because the sum

$$\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3} + \ldots \text{ is divergent.}$$

Hence the product is *zero* or *infinity* according as $m+1$ is *positive* or *negative*. (See Art. 462.)

520. It was shown in Arts. 416, 418 that $\frac{1}{2}(x^n + x^{-n})$ is the same function of $\frac{1}{2}(x + x^{-1})$ as $\cos na$ is of $\cos a$: and that $\frac{x^n - x^{-n}}{x - x^{-1}}$ is the same function of $\frac{1}{2}(x + x^{-1})$ as $\frac{\sin na}{\sin a}$ is of $\cos a$.

Put $x = e^a$; then $\frac{1}{2}(x + x^{-1}) = \frac{1}{2}(e^a + e^{-a}) = \cosh a$; $\frac{1}{2}(x - x^{-1}) = \sinh a$; $\frac{1}{2}(x^n + x^{-n}) = \cosh na$; $\frac{1}{2}(x^n - x^{-n}) = \sinh na$.

Thus $\cosh na$ is the same function of $\cosh a$ as $\cos na$ is of $\cos a$.

And $\frac{\sinh na}{\sinh a}$ is the same function of $\cosh a$ as $\frac{\sin na}{\sin a}$ is of $\cos a$.

Hence the second formulæ of Arts. 419, 420, 421 become

$$\cosh n\theta = 2^{n-1}\left(\cosh\theta - \cos\frac{\pi}{2n}\right)\left(\cosh\theta - \cos\frac{3\pi}{2n}\right)\ldots\left(\cosh\theta - \cos\frac{2n-1}{2n}\pi\right).$$

$$\sinh n\theta = 2^{n-1}\sinh\theta\left(\cosh\theta - \cos\frac{\pi}{n}\right)\ldots\left(\cosh\theta - \cos\frac{n-1}{n}\pi\right),$$

$$\cosh n\theta - \cos na = 2^{n-1}(\cosh\theta - \cos a)\left\{\cosh\theta - \cos\left(a + \frac{2\pi}{n}\right)\right\}$$
$$\ldots\left\{\cosh\theta - \cos\left(a + \frac{2n-2}{n}\pi\right)\right\}.$$

521. *To resolve* $\cos\theta$ *and* $\cosh\theta$ *into factors.*

We have $\cos na$ and $\cosh na$ equal respectively to

$$2^{n-1}\left(\cos a - \cos\frac{\pi}{2n}\right)\left(\cos a - \cos\frac{3\pi}{2n}\right)\ldots\left(\cos a - \cos\frac{2n-1}{2n}\pi\right),$$

$$2^{n-1}\left(\cosh a - \cos\frac{\pi}{2n}\right)\left(\cosh a - \cos\frac{3\pi}{2n}\right)\ldots\left(\cosh a - \cos\frac{2n-1}{2n}\pi\right).$$

Now $\cos(2n - \lambda)\pi/2n = -\cos\lambda\pi/2n$.

$$\therefore \left(\cos a - \cos\frac{\lambda\pi}{2n}\right)\left(\cos a - \cos\frac{2n-\lambda}{2n}\pi\right) = \sin^2\frac{\lambda\pi}{2n} - \sinh^2 a,$$

$$\left(\cosh a - \cos\frac{\lambda\pi}{2n}\right)\left(\cosh a - \cos\frac{2n-\lambda}{2n}\pi\right) = \sin^2\frac{\lambda\pi}{2n} + \sinh^2 a.$$

$\therefore$ if n is even, multiplying first and last factors &c., we have

$$2^{n-1}\left(\sin^2\frac{\pi}{2n}-\sin^2 a\right)\left(\sin^2\frac{3\pi}{2n}-\sin^2 a\right)\ldots\left(\sin^2\frac{n-1}{2n}\pi-\sin^2 a\right)$$

$$2^{n-1}\left(\sin^2\frac{\pi}{2n}+\sinh^2 a\right)\left(\sin^2\frac{3\pi}{2n}+\sinh^3 a\right)\ldots\left(\sin^2\frac{n-1}{2n}\pi+\sinh^2 a\right).$$

Here put $na=\theta$. Thus $\cos\theta$ and $\cosh\theta$ are equal respectively to

$$2^{n-1}\left(\sin^2\frac{\pi}{2n}-\sin^2\frac{\theta}{n}\right)\left(\sin^2\frac{3\pi}{2n}-\sin^2\frac{\theta}{n}\right)\ldots\left(\sin^2\frac{n-1}{2n}\pi-\sin^2\frac{\theta}{n}\right),$$

$$2^{n-1}\left(\sin^2\frac{\pi}{2n}+\sinh^2\frac{\theta}{n}\right)\left(\sin^2\frac{3\pi}{2n}+\sinh^2\frac{\theta}{n}\right)\ldots\left(\sin^2\frac{n-1}{2n}\pi+\sinh^2\frac{\theta}{n}\right).$$

Here put $\theta=0$. Thus

$$1=2^{n-1}\sin^2\frac{\pi}{2n}\sin^2\frac{3\pi}{2n}\ldots\sin^2\frac{n-1}{2n}\pi.$$

Divide each of the above two equations by this last. Thus

$$\cos\theta=\left(1-\frac{\sin^2\theta/n}{\sin^2\pi/2n}\right)\left(1-\frac{\sin^2\theta/n}{\sin^2 3\pi/2n}\right)\ldots\left(1-\frac{\sin^2\theta/n}{\sin^2(n-1)\pi/2n}\right),$$

$$\cosh\theta=\left(1+\frac{\sin^2\theta/n}{\sin^2\pi/2n}\right)\left(1+\frac{\sinh^2\theta/n}{\sin^2 3\pi/2n}\right)\ldots\left(1+\frac{\sinh^2\theta/n}{\sin^2(n-1)\pi/2n}\right).$$

In these last equations, make n infinite, while θ remains constant and finite. Then θ/n and $\lambda\pi/2n$ become zero: and, for the ratio of the functions sin and sinh we may substitute that of the quantities θ/n and $\lambda\pi/2n$ themselves. Thus

$$\cos\theta=\left\{1-\left(\frac{2\theta}{\pi}\right)^2\right\}\left\{1-\left(\frac{2\theta}{3\pi}\right)^2\right\}\left\{1-\left(\frac{2\theta}{5\pi}\right)^2\right\}\ldots \text{ to infinity.}$$

$$\cosh\theta=\left\{1+\left(\frac{2\theta}{\pi}\right)^2\right\}\left\{1+\left(\frac{2\theta}{3\pi}\right)^2\right\}\left\{1+\left(\frac{2\theta}{5\pi}\right)^2\right\}\ldots \text{ to infinity.}$$

These products are convergent for all finite values of θ, because the sum of the series

$$\left(\frac{2\theta}{\pi}\right)^2\left(\frac{1}{1}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right)$$

is convergent: the n^{th} term being of the form $(2n+1)^x$, where $x=-2<-1$.

The first formulæ may easily be remembered, by giving to θ the values $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ &c. which make each side vanish.

522. *To resolve* $\sin\theta$ *and* $\sinh\theta$ *into factors.*

We have

$$\sin na = 2^{n-1}\sin a\left(\cos a - \cos\frac{\pi}{n}\right)\ldots\left(\cos a - \cos\frac{n-1}{n}\pi\right),$$

$$\sinh na = 2^{n-1}\sinh a\left(\cosh a - \cos\frac{\pi}{n}\right)\ldots\left(\cosh a - \cos\frac{n-1}{n}\pi\right).$$

Now $\cos(n-\lambda)\pi/n = -\cos\lambda\pi/n$;

$$\therefore \left(\cos a - \cos\frac{\lambda\pi}{n}\right)\left(\cos a - \cos\frac{n-\lambda}{n}\pi\right) = \sin^2\frac{\lambda\pi}{n} - \sin^2 a,$$

$$\left(\cosh a - \cos\frac{\lambda\pi}{n}\right)\left(\cosh a - \cos\frac{n-\lambda}{n}\pi\right) = \sin^2\frac{\lambda\pi}{n} + \sinh^2 a;$$

$\therefore$ if n is odd, multiplying second and last factors &c.,

$$\sin na = 2^{n-1}\sin a\left(\sin^2\frac{\pi}{n} - \sin^2 a\right)\ldots\left(\sin^2\frac{n-1}{2n}\pi - \sin^2 a\right),$$

$$\sinh na = 2^{n-1}\sinh a\left(\sin^2\frac{\pi}{n} + \sinh^2 a\right)\ldots\left(\sin^2\frac{n-1}{2n}\pi + \sinh^2 a\right).$$

Here put $na = \theta$. Thus

$$\sin\theta = 2^{n-1}\sin\frac{\theta}{n}\left(\sin^2\frac{\pi}{n} - \sin^2\frac{\theta}{n}\right)\ldots\left(\sin^2\frac{n-1}{2n}\pi - \sin^2\frac{\theta}{n}\right),$$

$$\sinh\theta = 2^{n-1}\sinh\frac{\theta}{n}\left(\sin^2\frac{\pi}{n} + \sinh^2\frac{\theta}{n}\right)\ldots\left(\sin^2\frac{n-1}{2n}\pi + \sinh^2\frac{\theta}{n}\right).$$

Here put $\theta = 0$. Thus

$$n = 2^{n-1}\sin^2\frac{\pi}{n}\sin^2\frac{2\pi}{n}\ldots\sin^2\frac{n-1}{2n}\pi.$$

Divide each of the above two equations by this last. Thus

$$\sin\theta = n\sin\frac{\theta}{n}\cdot\left(1 - \frac{\sin^2\theta/n}{\sin^2\pi/n}\right)\ldots\left(1 - \frac{\sin^2\theta/n}{\sin^2(n-1)\pi/2n}\right),$$

$$\sinh\theta = n\sinh\frac{\theta}{n}\cdot\left(1 + \frac{\sinh^2\theta/n}{\sin^2\pi/n}\right)\ldots\left(1 + \frac{\sinh^2\theta/n}{\sin^2(n-1)\pi/2n}\right)$$

Here make n infinite. Thus

$$\sin\theta = \theta\left\{1-\left(\frac{\theta}{\pi}\right)^2\right\}\left\{1-\left(\frac{\theta}{2\pi}\right)^2\right\}\left(1-\left(\frac{\theta}{3\pi}\right)^2\right\}\text{....to infinity.}$$

$$\sinh\theta = \theta\left\{1+\left(\frac{\theta}{\pi}\right)^2\right\}\left\{1+\left(\frac{\theta}{2\pi}\right)^2\right\}\left\{1+\left(\frac{\theta}{3\pi}\right)^2\right\}\text{ ... to infinity.}$$

These products are also convergent.

The first formula may easily be remembered by giving to θ the values 0, π, 2π &c., which make each side vanish.

523. *To resolve* $\cos\theta - \cos\phi$ *and* $\cosh\theta - \cos\phi$ *into factors.*

We have

$$\cos n\alpha - \cos n\beta = 2^{n-1} P_{\lambda=0}^{\lambda=n-1}\left\{\cos\alpha - \cos\left(\frac{2\lambda\pi}{n}+\beta\right)\right\},$$

$$\cosh n\alpha - \cos n\beta = 2^{n-1} P_{\lambda=0}^{\lambda=n-1}\left\{\cosh\alpha - \cos\left(\frac{2\lambda\pi}{n}+\beta\right)\right\};$$

where P denotes the *product* of the factors obtained by giving to λ the integral values between the assigned limits.

Now $\cos(2\lambda\pi/n+\beta) = \cos\{2(n-\lambda)\pi/n-\beta\}$,

$\therefore$ if n is even, $\lambda = \frac{1}{2}n$ gives $\cos(2\lambda\pi/n+\beta) = -\cos\beta$, and

$\cos n\alpha - \cos n\beta$

$$= 2^{n-1}(\cos^2\alpha - \cos^2\beta) P_{\lambda=1}^{\lambda=\frac{n}{2}-1}\left\{\cos\alpha - \cos\left(\frac{2\lambda\pi}{n}\pm\beta\right)\right\},$$

$\cosh n\alpha - \cos n\beta$

$$= 2^{n-1}(\cosh^2\alpha - \cos^2\beta) P_{\lambda=1}^{\lambda=\frac{n}{2}-1}\left\{\cosh\alpha - \cos\left(\frac{2\lambda\pi}{n}\pm\beta\right)\right\}.$$

Again $\cos(2\lambda\pi/n \pm \beta) = -\cos\{(n-2\lambda)\pi/n \mp \beta\}$;

$$\therefore \left\{\cos\alpha - \cos\left(\frac{2\lambda\pi}{n}\pm\beta\right)\right\}\left\{\cos\alpha - \cos\left(\frac{n-2\lambda}{n}\pi\mp\beta\right)\right\}$$

$$= \sin^2\left(\frac{2\lambda\pi}{n}\pm\beta\right) - \sin^2\alpha,$$

$$\left\{\cosh\alpha - \cos\left(\frac{2\lambda\pi}{n}\pm\beta\right)\right\}\left\{\cosh\alpha - \cos\left(\frac{n-2\lambda}{n}\pi\mp\beta\right)\right\}$$

$$= \sin^2\left(\frac{2\lambda\pi}{n}\pm\beta\right) + \sinh^2\alpha.$$

If then $\frac{1}{2}n - 1$ is even, writing $n\alpha = \theta$ and $n\beta = \phi$, we have

$$\cos\theta - \cos\phi = 2^{n-1} \prod_{\lambda=0}^{\lambda=\frac{1}{2}(n-2)} \left\{\sin^2\left(\frac{2\lambda\pi}{n} \pm \frac{\phi}{n}\right) - \sin^2\frac{\theta}{n}\right\},$$

$$\cosh\theta - \cos\phi = 2^{n-1} \prod_{\lambda=0}^{\lambda=\frac{1}{2}(n-2)} \left\{\sin^2\left(\frac{2\lambda\pi}{n} \pm \frac{\phi}{n}\right) + \sinh^2\frac{\theta}{n}\right\}.$$

Here put $\theta = 0$. Thus

$$1 - \cos\phi = 2^{n-1} \prod_{\lambda=0}^{\lambda=\frac{1}{2}(n-2)} \sin^2\left(\frac{2\lambda\pi}{n} \pm \frac{\phi}{n}\right).$$

Divide each of the above two equations by this last. Thus

$$\cos\theta - \cos\phi = 2\sin^2\frac{\phi}{2}\left(1 - \frac{\sin^2\theta/n}{\sin^2\phi/n}\right)\left(1 - \frac{\sin^2\theta/n}{\sin^2(2\pi \pm \phi)/n}\right)\ldots$$

$$\cosh\theta - \cos\phi = 2\sin^2\frac{\phi}{2}\left(1 + \frac{\sinh^2\theta/n}{\sin^2\phi/n}\right)\left(1 + \frac{\sinh^2\theta/n}{\sin^2(2\pi \pm \phi)/n}\right)\ldots$$

Here make n infinite. Thus

$$\cos\theta - \cos\phi = 2\sin^2\frac{\phi}{2}\left\{1 - \left(\frac{\theta}{\phi}\right)^2\right\}\left\{1 - \left(\frac{\theta}{2\pi \pm \phi}\right)^2\right\}\left\{1 - \left(\frac{\theta}{4\pi \pm \phi}\right)^2\right\}\ldots$$

$$\cosh\theta - \cos\phi = 2\sin^2\frac{\phi}{2}\left\{1 + \left(\frac{\theta}{\phi}\right)^2\right\}\left\{1 + \left(\frac{\theta}{2\pi \pm \phi}\right)^2\right\}\left\{1 + \left(\frac{\theta}{4\pi \pm \phi}\right)^2\right\}\ldots$$

These products are clearly convergent.

The first formula may be easily remembered by giving to θ the values ϕ, $2\pi \pm \phi$, $4\pi \pm \phi$ &c. which make both sides vanish; and by putting $\theta = 0$ which gives $1 - \cos\phi = 2\sin^2\frac{1}{2}\phi$.

524. Many important results may be obtained from the factorisation of $\cos\theta$ and $\sin\theta$.

Thus we may equate the coefficients of the several powers of θ in the term-expansions of $\cos\theta$ and $\sin\theta$ with those obtained by multiplying out the factor expressions.

Or we may equate coefficients after taking the logarithms of the two expressions for $\cos\theta$ or $\sin\theta$.

Thus we have, from the two expressions for $\sin\theta$,

$$1 - \frac{\theta^2}{\lfloor 3} + \frac{\theta^4}{\lfloor 5} - \ldots = \left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\ldots$$

Equating the coefficients of θ^2,

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots \text{ ad inf.} = \frac{\pi^2}{6}.$$

525. The method of the following article is of very general application, the process involved being in reality merely to *differentiate* both sides of an identity.

526. *To find series for* cot θ *and for* tan θ.

Resolving each factor of sin θ into two factors, we may write

$$\sin\theta = \theta P\left(1+\frac{\theta}{r\pi}\right),$$

where P denotes the product of all values obtained by giving to r every positive and negative integral value, *excluding zero.*

$$\therefore \qquad \log\sin\theta = \log\theta + \Sigma\log\left(1+\frac{\theta}{r\pi}\right).$$

Now for θ write $\theta+h$;

$$\therefore \qquad \log\sin(\theta+h) = \log(\theta+h) + \Sigma\log\left(1+\frac{\theta+h}{r\pi}\right).$$

Subtract the identity in θ *from the corresponding identity in* $\theta+\mathrm{h}$; *and expand the remainder in powers of* h.

On the left we have

$$\begin{aligned}\log\sin(\theta+h)-\log\sin\theta &= \log(\cos h+\cot\theta\sin h)\\ &= \log(1+h\cot\theta-\tfrac{1}{2}h^2\ldots)\\ &= h\cot\theta-\tfrac{1}{2}h^2-\ldots-\tfrac{1}{2}(h\cot\theta\ldots)^2+\ldots.\end{aligned}$$

On the right we have

$$\log(\theta+h)-\log\theta = \log\left(1+\frac{h}{\theta}\right) = \frac{h}{\theta}-\frac{1}{2}\frac{h^2}{\theta^2}+\ldots$$

$$\begin{aligned}\log\left(1+\frac{\theta+h}{r\pi}\right)-\log\left(1+\frac{\theta}{r\pi}\right) &= \log\left(1+\frac{h}{r\pi+\theta}\right)\\ &= \frac{h}{r\pi+\theta}-\frac{1}{2}\frac{h^2}{(r\pi+\theta)^2}+\ldots.\end{aligned}$$

Hence equating the coefficients of h in the new identity,

$$\cot\theta = \Sigma\frac{1}{r\pi+\theta},$$

where Σ denotes the sum of all values obtained by giving to r every positive and negative integral value *including zero.*

In the same way from the identity

$$\cos\theta = P\left(1 + \frac{\theta}{(2r+1)\cdot\frac{1}{2}\pi}\right)$$

it follows that

$$-\tan\theta = \Sigma\,\frac{1}{(2r+1)\,\frac{1}{2}\pi + \theta},$$

where, as above, r has every value *including zero.*

In these expansions of cot θ and tan θ, the terms having equal and opposite coefficients of π should be added so as to form one term. [Otherwise the series assumes the indeterminate form $\infty - \infty$.] Thus

$$\cot\theta = \frac{1}{\theta} + \Sigma_1^{\infty}\,\frac{2\theta}{\theta^2 - r^2\pi^2},$$

$$\tan\theta = \Sigma_0^{\infty}\,\frac{2\theta}{\{(2r+1)\,\frac{1}{2}\pi\}^2 - \theta^2}.$$

§ 4. Circular Measure in Terms of Ratios.

527. *To express θ in terms of* tan θ.

We have shown in Art. 359 that for all positive integral values of n,

$$n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{1.2.3}\cos^{n-3}\theta\sin^3\theta + \ldots = \sin n\theta.$$

Divide by $n\cos^n\theta$. Then

$$\tan\theta - \frac{(n-1)(n-2)}{1.2.3}\tan^3\theta + \frac{(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}\tan^5\theta - \ldots = \frac{\sin n\theta}{n\cos^n\theta}.$$

If now n is *not* a positive integer the series on the left will be endless. If moreover tan θ is arithmetically equal to or less than 1, the series will be *convergent* as in the binomial theorem.

Now in this case the above equation—as will be shown in Chap. XXI.—still holds, provided that, on the right hand, we make θ represent the positive or negative *acute* angle whose tangent = tan θ; and give to $\cos^n\theta$ its *positive* value.

With this understanding, then, we may make n *as small a fraction as we please.* Then, in the limit,

$$\frac{\sin n\theta}{n\cos^n\theta} \equiv \theta . \frac{\sin n\theta}{n\theta} . \frac{1}{(\cos\theta)^n} = \theta.$$

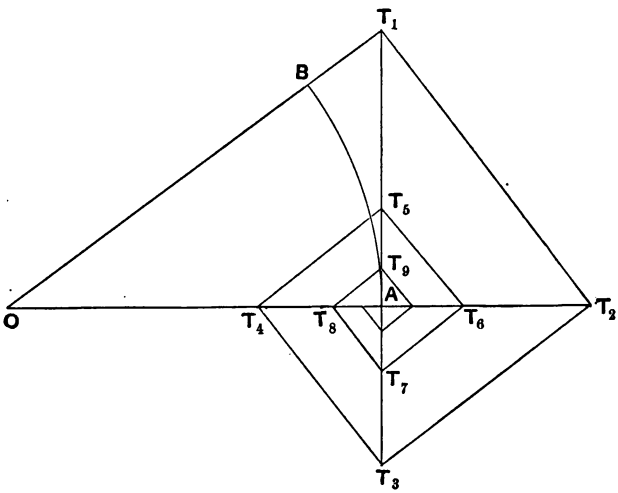
Hence from the above equation,

If $\tan\theta \gtreqless 1$ and θ is acute; i.e. if θ lies between $-\frac{1}{4}\pi$ and $+\frac{1}{4}\pi$,

$$\tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \ldots = \theta.$$

This is called *Gregory's* series.

528. *The geometrical illustration of Gregory's series.*



Let AOB be any angle not greater than half a right-angle; AB the arc opposite AOB; AT_1 the tangent at A.

Draw the lines OBT_1, T_1T_2, T_2T_3, T_3T_4,... each perpendicular to the preceding to meet AT_1 and OA alternately. Then

$$\text{arc } AB = AT_1 - \tfrac{1}{3}AT_3 + \tfrac{1}{5}AT_5 - \tfrac{1}{7}AT_7 + \ldots$$

For, if $AO = 1$, $AB =$ circular measure of $AB = \theta$ say.

And $AT_1 = \tan\theta$; $AT_2 = \tan^2\theta$; $AT_3 = \tan^3\theta$; and so on, by Euclid VI. 8.

529. *To express θ in terms of sin θ.*

We have shown in Art. 375, that for all *odd* integral values of n,

$$\sin\theta - \frac{n^2-1^2}{\lfloor 3}\sin^3\theta + \frac{(n^2-1^2)(n^2-3^2)}{\lfloor 5}\sin^5\theta - \ldots = \frac{\sin n\theta}{n}.$$

If now n is *not* an odd integer the series on the left will be endless. It is always convergent, because $\sin\theta$ is arithmetically equal to or less than 1.

Now the above equation—as will be shown in Chap. XXI.—still holds provided that, on the right hand, we make θ represent the positive or negative *acute* angle whose sine $= \sin\theta$.

Thus we may make n as small a fraction as we please. Then in the limit, if θ is acute,

$$\sin\theta + \frac{1}{2}\cdot\frac{\sin^3\theta}{3} + \frac{1.3}{2.4}\cdot\frac{\sin^5\theta}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{\sin^7\theta}{7} + \ldots = \theta.$$

530. If x is any quantity arithmetically not greater than 1, in Art. 527, we may write $\tan\theta = x$, and in Art. 529, $\sin\theta = x$. Thus

$x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots =$ *circular* measure of the *acute* angle whose *tangent* is equal to x.

$$x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + \ldots =$$ *circular* measure of the *acute* angle whose *sine* is equal to x.

531. Now we have $\tan(r\pi+\theta) = \tan\theta$ where r is any integer. If then ϕ is any angle lying between $r\pi - \frac{1}{4}\pi$ and $r\pi + \frac{1}{4}\pi$, *where* r *has some assigned integral value,*

$$\tan\phi - \tfrac{1}{3}\tan^3\phi + \tfrac{1}{5}\tan^5\phi - \ldots = \phi - r\pi.$$

Again we have $\cot(r\pi + \frac{1}{2}\pi - \theta) = \tan\theta$, where r is any integer. If then ϕ is any angle lying between $r\pi + \frac{1}{4}\pi$ and $r\pi + \frac{3}{4}\pi$, *where* r *has some assigned integral value,*

$$\cot\phi - \tfrac{1}{3}\cot^3\phi + \tfrac{1}{5}\cot^5\phi - \ldots = r\pi + \tfrac{1}{2}\pi - \phi.$$

Hence ϕ may be expanded in powers of $\tan\phi$, when ϕ is in the *right* or *left* quadrant, and in powers of $\cot\phi$ when it is in the *up* or *down* quadrant.

532. Now we have $\sin\{r\pi+(-1)^r\theta\}=\sin\theta$. If then ϕ is any angle lying between $r\pi-\frac{1}{2}\pi$ and $r\pi+\frac{1}{2}\pi$, *where* r *has some assigned integral value*,

$$\sin\phi+\frac{1}{2}\cdot\frac{\sin^3\phi}{3}+\frac{1.3}{2.4}\cdot\frac{\sin^5\phi}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{\sin^7\phi}{7}+\ldots=(-1)^r(\phi-r\pi).$$

Writing here $\phi=\frac{1}{2}\pi+\psi$, we have if ψ is any angle lying between $(r-1)\pi$ and $r\pi$, *where* r *has some assigned integral value*,

$$\cos\psi+\frac{1}{2}\cdot\frac{\cos^3\psi}{3}+\frac{1.3}{2.4}\cdot\frac{\cos^5\psi}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{\cos^7\psi}{7}+\ldots$$
$$=(-1)^r(\psi+\tfrac{1}{2}\pi-r\pi).$$

Hence *any* angle ψ may be expanded in powers of $\sin\psi$ or of $\cos\psi$.

The calculation of π.

533. The two series of Art. 529 enable us to calculate the value of an arc in terms of its sine or tangent. In particular we may find from them the ratio of the circumference to the diameter of a circle; i.e. the value of π.

Using the inverse symbol to represent the *circular* measure of an *acute* angle, we have

$$\sin^{-1}x=x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\cdot\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7}+\ldots$$

$$\tan^{-1}x=x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\tfrac{1}{7}x^7+\ldots$$

534. In the first of the above equations, let $x=\frac{1}{2}$. Then $\sin^{-1}x=\frac{1}{6}\pi$. Thus

$$\frac{\pi}{6}=\frac{1}{2}+\frac{1}{3.2^4}+\frac{3}{2}\cdot\frac{1}{5.2^7}+\frac{3.5}{\underline{|3}}\cdot\frac{1}{7.2^{10}}+\frac{3.5.7}{\underline{|4}}\cdot\frac{1}{9.2^{13}}+\ldots$$

$$\therefore\quad \pi=3+\frac{1}{8}+\frac{9}{10.2^6}+\frac{3.5}{7.2^{10}}+\frac{3.5.7}{9.2^{15}}+\ldots$$

Taking 3 terms of this series, we have $\pi=3{\cdot}14$ approximately.

After the first few terms, this series does not converge rapidly enough for convenience of calculation.

535. In the second of the equations of Art. 533, put $x = 1$. Then $\tan^{-1} x = \frac{1}{4}\pi$. Thus

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$$

$$\therefore \quad \frac{\pi}{8} = \frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \dots$$

This series is very slowly convergent.

We may however adapt the inverse-tangent series for the purposes of calculation by expressing $\frac{1}{4}\pi$ as the sum or difference of suitable angles.

The excellence of a series for purposes of calculation depends on (1) the rapidity of its convergence, and (2) the simplicity of the operations to be performed.

536. We shall make use of the equation

$$\tan^{-1} T = \tan^{-1} t + \tan^{-1} \frac{T - t}{1 + Tt}.$$

Here T and t may be chosen at will, and we shall attempt to find simple expressions for the remaining term.

537. *First*, put $t = \frac{1}{2}$.

Then if $T = 1$, we have

$$\frac{1}{4}\pi = \tan^{-1} \tfrac{1}{2} + \tan^{-1} \tfrac{1}{3} \dots\dots\dots\dots\dots\dots (1).$$

And if $T = \frac{1}{3}$, we have

$$\tan^{-1} \tfrac{1}{3} = \tan^{-1} \tfrac{1}{2} - \tan^{-1} \tfrac{1}{7}.$$

Substituting in (1) for $\tan^{-1} \frac{1}{3}$ or $\tan^{-1} \frac{1}{2}$ from this last equation we have

$$\frac{1}{4}\pi = 2 \tan^{-1} \tfrac{1}{2} - \tan^{-1} \tfrac{1}{7} \dots\dots\dots\dots\dots (2),$$

$$\frac{1}{4}\pi = 2 \tan^{-1} \tfrac{1}{3} + \tan^{-1} \tfrac{1}{7} \dots\dots\dots\dots\dots (3)$$

538. *Next*, put $t = \frac{1}{4}$.

If $T = 1$; $\quad \frac{1}{4}\pi = \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{3}{5}$.

If $T = \frac{3}{5}$; $\quad \tan^{-1} \frac{3}{5} = \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{7}{23}$.

If $T = \frac{7}{23}$; $\quad \tan^{-1} \frac{7}{23} = \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{5}{99}$.

$\therefore$ (adding) $\quad \frac{1}{4}\pi = 3 \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{5}{99} \dots\dots\dots\dots\dots (4).$

This is convenient, since 99 has the simple divisors 9 and 11.

539. *Next*, put $t=\frac{1}{5}$.

If $T=1$; $\quad \frac{1}{4}\pi=\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{2}{3}$.

If $T=\frac{2}{3}$; $\quad \tan^{-1}\frac{2}{3}=\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{7}{17}$.

If $T=\frac{7}{17}$; $\quad \tan^{-1}\frac{7}{17}=\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{9}{46}$.

If $T=\frac{9}{46}$; $\quad \tan^{-1}\frac{9}{46}=\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}$.

$\therefore$ (adding) $\quad \frac{1}{4}\pi=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}$ .....(5).

This series is quickly converging, but the divisor 239 is inconveniently large.

540. We have thus found *five* substitutions for $\frac{1}{4}\pi$ by means of which its value may be calculated more or less conveniently. Thus

$$\frac{1}{4}\pi=\frac{1}{2}\times\left(1-\frac{1}{3.4}+\frac{1}{5.4^2}-\frac{1}{7.4^3}+\ldots\right)$$
$$+\frac{1}{3}\times\left(1-\frac{1}{3.9}+\frac{1}{5.9^2}-\frac{1}{7.9^3}+\ldots\right) \ldots\ldots\ldots\ldots(1).$$

$$\frac{1}{4}\pi=\quad 1-\frac{1}{3.4}+\frac{1}{5.4^2}-\frac{1}{7.4^3}+\ldots$$
$$-\frac{1}{7}\times\left(1-\frac{1}{3.49}+\frac{1}{5.49^2}-\frac{1}{7.49^3}+\ldots\right) \ldots\ldots\ldots(2).$$

$$\frac{1}{4}\pi=\frac{2}{3}\times\left(1-\frac{1}{3.9}+\frac{1}{5.9^2}-\frac{1}{7.9^3}+\ldots\right)$$
$$+\frac{1}{7}\times\left(1-\frac{1}{3.49}+\frac{1}{5.49^2}-\frac{1}{7.49^3}+\ldots\right)\ldots\ldots\ldots (3).$$

$$\frac{1}{4}\pi=\frac{3}{4}\times\left(1-\frac{1}{3.16}+\frac{1}{5.16^2}-\frac{1}{7.16^3}+\ldots\right)$$
$$+\frac{5}{99}\times\left(1-\frac{5^2}{3.99^2}+\frac{5^4}{5.99^4}-\frac{5^6}{7.99^6}+\ldots\right)\ldots\ldots (4).$$

$$\frac{1}{4}\pi=\frac{4}{5}\times\left(1-\frac{1}{3.5^2}+\frac{1}{5.5^4}-\frac{1}{7.5^6}+\ldots\right)$$
$$-\frac{1}{239}\times\left(1-\frac{1}{3.239^2}+\frac{1}{5.239^4}-\frac{1}{7.239^6}+\ldots\right) (5).$$

Series (1) is known as *Euler's;* series (5) as *Machin's;* the other series were suggested by *Hutton*, whose method has been followed in the above derivations.

The last series may be simplified by noting that

$$\tan^{-1}\tfrac{1}{239} = \tan^{-1}\tfrac{1}{70} - \tan^{-1}\tfrac{1}{99},$$

which the student should prove.

541. We will now prove the important proposition that π and also π^2 are incommensurable. For this purpose it will be necessary to give a short account of *continued fractions*.

Continued Fractions.

542. A continued fraction is a fraction of the form

$$\cfrac{b_1}{a_1 \pm \cfrac{b_2}{a_2 \pm \&c.}}$$ and is usually written shortly $\frac{b_1}{a_1} \pm \frac{b_2}{a_2} \pm \frac{b_3}{a_3} \pm \dots$.

We will give a few theorems on fractions of the form

$$\frac{b_1}{a_1} - \frac{b_2}{a_2} - \frac{b_3}{a_3} -, \text{ where } a_1,\ b_1,\ a_2,\ b_2 \dots$$

are all positive. Such continued fractions are said to be of the *second class*.

The simple fractions $\frac{b_1}{a_1}, \frac{b_2}{a_2}, \frac{b_3}{a_3}$ are called the *components*: the fractions obtained by *stopping* at these are called the *convergents*: and the fractions obtained by *omitting* them are called the *remainders*.

Thus $\frac{b_n}{a_n}$ being the n^{th} component, $\frac{b_{n+1}}{a_{n+1}} - \frac{b_{n+2}}{a_{n+2}} - \dots$ is the n^{th} remainder.

The first convergent is $\frac{b_1}{a_1}$; the second $\cfrac{b_1}{a_1 - \cfrac{b_2}{a_2}} = \frac{a_2 b_1}{a_1 a_2 - b_2}$, and so on.

Also, if F is the fraction, and ρ_n the n^{th} remainder,

$$F = \frac{b_1}{a_1 - \rho_1};\ \rho_1 = \frac{b_2}{a_2 - \rho_2}, \text{ and generally } \rho_{n-1} = \frac{b_n}{a_n - \rho_n}.$$

Lastly, if c_n is the n^{th} convergent of F, and γ_n the n^{th} convergent of ρ_1,

$$c_{n+1} = \frac{b_1}{a_1 - \gamma_n}.$$

543. Theorem I. The convergents are in ascending order of magnitude, provided that they are all positive.

For we have $c_n = \dfrac{b_1}{a_1 - \gamma_{n-1}}$ and $c_{n+1} = \dfrac{b_1}{a_1 - \gamma_n}$.

Assuming, then, that a convergent γ_n of the n^{th} order is greater than γ_{n-1} of the $(n-1)^{\text{th}}$ order, we see that c_{n+1} of the $(n+1)^{\text{th}}$ order is greater than c_n of the n^{th} order.

Now clearly the 2nd convergent $\dfrac{b_1}{a_1 - b_2/a_2}$ is greater than the 1st convergent $\dfrac{b_1}{a_1}$. Hence, universally, a convergent is increased by taking an additional component.

544. Theorem II. If the denominator of every component exceeds the numerator by unity at least, the continued fraction is itself positive and not greater than unity.

For assuming that γ_n of the n^{th} order is not greater than unity, c_{n+1} of the $(n+1)^{\text{th}}$ order is at greatest $\dfrac{b_1}{a_1 - 1}$, which is not greater than unity; and is positive, since a_1 is not less than γ_n.

Now clearly the 1st convergent $\dfrac{b_1}{a_1}$ is not greater than unity. Hence *no* convergent can be greater than unity; and all are positive.

And, therefore, the fraction itself is positive but not greater than unity.

545. Theorem III. If the denominator of every component exceeds the numerator by unity at least, while that of the first exceeds its numerator by more than unity, the continued fraction is itself less than unity.

For the fraction $= \dfrac{b_1}{a_1 - \rho_1}$, where, by the last theorem, ρ_1 is not greater than unity. Hence the fraction is at greatest $\dfrac{b_1}{a_1 - 1}$ which is less than unity.

546. Theorem IV. An endless continued fraction of the second class, in which the numerator and denominator of every component are finite integers, and in which every remainder is positive and less than unity, must be incommensurable.

For, if possible, let the fraction be commensurable and equal to $\dfrac{B}{A}$ where A and B are finite positive integers.

Then $\dfrac{B}{A} = \dfrac{b_1}{a_1 - \rho_1}$, $\quad \therefore \rho_1 = \dfrac{a_1 B - b_1 A}{B} = \dfrac{C}{B}$ say.

Here C the numerator of ρ_1 is a finite positive integer.

Similarly $\rho_2 = \dfrac{D}{C}$; $\quad \rho_3 = \dfrac{E}{D}$, &c., where D, E are positive integers.

Now, by hypothesis, each of the remainders ρ_1, ρ_2 ... is less than unity. Hence B, C, D, E... form a series of positive integers which are in descending order of magnitude and yet infinite in number: this is absurd. Hence the endless continued fraction cannot be commensurable.

547. Since the convergents of a continued fraction of the second class (if positive) are in ascending order, such a fraction resembles a series *all of whose terms are positive.* Hence moreover such a fraction cannot have more than one finite limit.

It should be observed that the assigned characteristics may be supposed to begin to hold after any *finite* number of components. For example, the n^{th} convergent of *any* continued fraction must be commensurable if its components are commensurable. Hence, if the remainder after the n^{th} convergent is incommensurable, the whole fraction is incommensurable.

548. *To prove that π is incommensurable.*

By Art. 513,

$$\tan\theta = \frac{\theta}{1-}\ \frac{\theta^2}{3-}\ \frac{\theta^2}{5-}\cdots$$

If possible, let $\frac{\pi}{4} = \frac{m}{n}$, where m and n are finite integers.

Put $\theta = \frac{1}{4}\pi$. Then $\tan\theta = 1$; thus

$$1 = \frac{m/n}{1-}\ \frac{m^2/n^2}{3-}\ \frac{m^2/n^2}{5-}\cdots$$

$$= \frac{m}{n-}\ \frac{m^2}{3n-}\ \frac{m^2}{5n-}\cdots$$

If r is taken large enough, in every component from $\frac{m^2}{(2r+1)\,n}$ onwards, the denominator will exceed the numerator by more than unity. Hence, by Theorem III., every remainder from this point onwards will be positive and less than unity. Hence, by Theorem IV., the fraction must be incommensurable. But it is not Hence π cannot be commensurable.

549. *To prove that π^2 is incommensurable.*

By Art. 513,

$$1 - \theta\cot\theta = \frac{\theta^2}{3-}\ \frac{\theta^2}{5-}\ \frac{\theta^2}{7-}\cdots$$

If possible let $\frac{\pi^2}{4} = \frac{m}{n}$, where m and n are finite integers.

Put $\theta = \frac{1}{2}\pi$. Then $\cot\theta = 0$; thus

$$1 = \frac{m/n}{3-}\ \frac{m/n}{5-}\ \frac{m/n}{7-}\cdots$$

$$= \frac{m}{3n-}\ \frac{mn}{5n-}\ \frac{mn}{7n-}\cdots$$

As before this is impossible. Hence π^2 cannot be commensurable.

Observe. This proposition includes the last.

EXAMPLES XIX.

1. $\tan\theta > \theta$ and $< \dfrac{2\theta}{2-\theta^2}$. 2. $\cot\theta < \dfrac{1}{\theta}$ and $> \dfrac{1}{\theta} - \dfrac{\theta}{2}$.

3. Given $\tan\theta > \theta$ and $\cos\theta > 1 - \frac{1}{2}\theta^2$, show that
$$\sin\theta > \theta - \tfrac{11}{54}\theta^3.$$

4. Given $\tan\theta > \theta$, show that $\cot\theta < \dfrac{1}{\theta} - \dfrac{\theta}{4}$; and further that $\cot\theta < \dfrac{1}{\theta} - \dfrac{\theta}{3}$.

5. If $x > \frac{1}{2}$, $\cot \dfrac{1}{1+x^2} > 1 - x + x^2$.

6. $\sin\theta > \tan\theta - \frac{1}{2}\tan^3\theta$.

7. $(2\cos\theta - 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1)$
$$\ldots \text{ to } n \text{ factors} = \frac{2\cos 2^n\theta + 1}{2\cos\theta + 1}.$$

8. $\left(1 - \tan^2\dfrac{\theta}{2}\right)\left(1 - \tan^2\dfrac{\theta}{2^2}\right)\left(1 - \tan^2\dfrac{\theta}{2^3}\right) \ldots \textit{ad inf.} = \dfrac{\theta}{\tan\theta}$.

9. $\frac{1}{2}\tan\frac{1}{2}\theta + \frac{1}{4}\tan\frac{1}{4}\theta + \frac{1}{8}\tan\frac{1}{8}\theta + \ldots \textit{ad inf.} = 1/\theta - \cot\theta$.

10. $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \ldots + \tan^{-1}\dfrac{1}{n^2+n+1} + \ldots \textit{ad inf.} = \dfrac{\pi}{4}$.

11. Find the limits, when x is zero, of
$(\cos x)^{\cot x}$, $(1 - \sin 2x)^{\operatorname{cosec} x}$, $(\cos ax)^{\operatorname{cosec}^2 \beta x}$, $(\sec ax)^{\cot^2 \beta x}$.

12. $\dfrac{2+\sqrt{3}}{4} \cdot \dfrac{2+\sqrt{(2+\sqrt{3})}}{4} \cdot \dfrac{2+\sqrt{\{2+\sqrt{(2+\sqrt{3})}\}}}{4}$
$$\ldots \textit{ad inf.} = \frac{9}{\pi^2}.$$

13. If θ measures an acute angle, $\theta > \dfrac{3\sin\theta}{2+\cos\theta}$.

14. If $\dfrac{\cos\theta}{\theta} + \dfrac{\theta}{\cos\theta}$ has its least positive value, $\theta > \sqrt{3} - 1$.

15. The chord of a circle subtending an angle θ at the centre $= 2 \sin \frac{1}{2}\theta$.

16. Without using any trigonometrical formula, but assuming only that the chord of a circle changes its sign without changing its magnitude when the arc changes its sign without changing its magnitude, and that the chord is nearly equal to the arc when both are small, prove the following three propositions for approximating to the value of the arc :—

(1) Subtract the chord of the arc from 8 times the chord of the half-arc and divide the result by 3.

(2) Add the chord of the arc to 256 times the chord of the quarter-arc, subtract 40 times the chord of the half-arc, and divide the result by 45.

(3) If

$$(4x-1)(4^2x-1)\ldots(4^nx-1) \equiv \lambda x^n - \mu x^{n-1} + \nu x^{n-2} - \ldots + (-1)^n$$

then

$$(4-1)(4^2-1)\ldots(4^n-1)\,a = \lambda \,.\, 2^n c_n - \mu \,.\, 2^{n-1} c_{n-1} + \ldots + (-1)^n c_0$$

where a = the arc, and c_m = the chord of $a/2^m$.

17. Find the limit, when $x = 1$, of $\dfrac{\log_e x}{x-1}$.

Find the following limits, when $x = 0$; or $\theta = 0$; (18—33).

18. $\dfrac{a^x - b^x}{x}$. 19. $\dfrac{x - \sin x}{x^3}$. 20. $\dfrac{\sin^2 px}{1 - \cos qx}$.

21. $\dfrac{\sinh x - x}{x - \sin x}$. 22. $\dfrac{\cosh x - 1}{1 - \cos x}$. 23. $\dfrac{\cosh x - \cos x}{\sinh x}$.

24. $\theta^{-5}(\sin\theta + \sinh\theta - 2\theta)$. 25. $\theta^{-6}(\cosh\theta - \cos\theta + \theta^2)$.

26. $\dfrac{m \sin x - \sin mx}{x(\cos x - \cos mx)}$. 27. $\dfrac{\sin^2 nx - \sin^2 mx}{1 - \cos px}$.

28. $\dfrac{2 \sin 3x - 3 \sin 2x}{x - \sin x}$. 29. $\dfrac{\sin 2x + 2 \sin^2 x - 2 \sin x}{\cos x - \cos^2 x}$.

30. $\dfrac{(\sin x + \sin 2x - \sin 3x)^2}{(\cos x - 2 \cos 2x + \cos 3x)^3}$.

31. $\dfrac{(5\sin x - 4\sin 2x + \sin 3x)^4}{(5\cos x - 8\cos 2x + 3\cos 3x)^5}$.

32. $\left(\dfrac{\tan\theta}{\theta}\right)^{\frac{3}{\theta^2}}$. 33. $\left(\cos\dfrac{\theta}{n} + \sin\dfrac{3\theta}{n}\right)^{\frac{n}{\theta}}$.

Prove the following statements: (34—49).

34. $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$.

35. $\sinh 2x = 2\sinh x \cosh x$.

36. $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$.

37. $\tanh x = \operatorname{cosech} 2x - \coth 2x$.

38. $\coth x = \operatorname{cosech} 2x + \coth 2x$.

39. $\cosh 3x = 4\cosh^3 x - 3\cosh x$.

40. $\sinh 3x = 4\sinh^3 x + 3\sinh x$.

41. The terms in the expansion of $\cosh nx$ and of $\sinh nx$ are taken alternately from the expansion of $(\cosh x + \sinh x)^n$.

42. $\operatorname{sech}^2 x + \tanh^2 x = 1$. 43. $\operatorname{cosech}^2 x + 1 = \coth^2 x$.

44. Cosh and Coth are always *greater* than 1; Sech and Tanh are always *less* than 1; Sinh and Cosech may have any value.

45. $\cosh y \operatorname{cosec} x + \sinh y \cot x > 1$.

46. If $\cos x = \operatorname{sech} y$; then $\sin x = \pm\tanh y$ and $\tan x = \pm\sinh y$.

47. If $\sin x = \operatorname{sech} y$; then $\cos x = \pm\tanh y$ and $\cot x = \pm\sinh y$.

48. $\cosh\pi = \left(1 + \dfrac{2^2}{1^2}\right)\left(1 + \dfrac{2^2}{3^2}\right)\left(1 + \dfrac{3^2}{5^2}\right) \ldots$ *ad inf.*

49. $\theta = \tanh\theta + \frac{1}{3}\tanh^3\theta + \frac{1}{5}\tanh^5\theta + \ldots$ *ad inf.*

Sum the following series to infinity: (50—55).

50. $x\sinh a + x^2\sinh 2a + x^3\sinh 3a + \ldots$

51. $x\cosh a + \dfrac{x^2}{\lfloor 2}\cosh 2a + \dfrac{x^3}{\lfloor 3}\cosh 3a + \ldots$

52. $1 - \frac{\cos a}{\lfloor 1} \cosh \beta + \frac{\cos^2 a}{\lfloor 2} \cosh 2\beta - \frac{\cos^3 a}{\lfloor 3} \cosh 3\beta + \ldots$

53. $\sinh a - \frac{1}{2} \sinh 2a + \frac{1}{3} \sinh 3a - \ldots$

54. $\cos\theta \cosh\phi + \frac{1}{2} \cos^2\theta \cosh 2\phi + \frac{1}{3} \cos^3\theta \cosh 3\phi + \ldots$

55. $1 + \frac{p}{q} \cos a \sinh \beta + \frac{p(p-q)}{\lfloor 2 \cdot q^2} \cos^2 a \sinh 2\beta + \frac{p(p-q)(p-2q)}{\lfloor 3 \cdot q^3} \cos^3 a \sinh 3\beta + \ldots$

Prove the following statements: (56—65).

56. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}$.

57. $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}$.

58. $\frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \ldots = \frac{\pi^2}{9}$.

59. The sum of the reciprocals of the squares of all integers except the multiples of r is $(r^2 - 1)\,\pi^2/6r^2$.

60. The sum of the reciprocals of the squares of all *odd* integers except the multiples of $2r+1$ is $\frac{r(r+1)\pi^2}{2(2r+1)^2}$.

61. The sum of the reciprocals of the squares of the products of all pairs of integers is $\frac{1}{120}\pi^4$; and of all pairs of *odd* integers is $\frac{1}{384}\pi^2$.

62. $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots = \frac{\pi^4}{90}$.

63. $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \ldots = \frac{\pi^4}{96}$.

64. $\frac{\pi}{4} = \frac{4^2}{3 \cdot 5} \cdot \frac{8^2}{7 \cdot 9} \cdot \frac{12^2}{11 \cdot 13} \cdots$

65. $\frac{\pi}{2} = \frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \frac{6^2}{5 \cdot 7} \cdots$

66. Show that $\dfrac{\sin(\theta+a)}{\sin\theta} = P\left(1+\dfrac{a}{r\pi+\theta}\right)$ and hence that

$$\cot\theta = \Sigma\,\frac{1}{r\pi+\theta} \text{ and } \operatorname{cosec}^2\theta = \Sigma\left(\frac{1}{r\pi+\theta}\right)^2.$$

67. Show that

$$\frac{\cos(\theta+a)}{\cos\theta} = P\left(1+\frac{a}{(2r+1)\,\frac{1}{2}\pi+\theta}\right)$$

and hence that

$$-\tan\theta = \Sigma\,\frac{1}{(2r+1)\,\frac{1}{2}\pi+\theta} \text{ and } \sec^2\theta = \Sigma\left(\frac{1}{(2r+1)\,\frac{1}{2}\pi+\theta}\right)^2.$$

68. Hence, using the identity, $\operatorname{cosec}\theta = \frac{1}{2}\cot\frac{1}{2}\theta + \frac{1}{2}\tan\frac{1}{2}\theta$, show that

$$\operatorname{cosec}\theta = \Sigma\,(-1)^r\frac{1}{r\pi+\theta},$$

and

$$\sec\theta = \Sigma\,(-1)^r\frac{1}{(2r+1)\,\frac{1}{2}\pi+\theta}.$$

69. If $\cos x\cosh x = 1$, then $\dfrac{\sin x-\sinh x}{\cos x-\cosh x} = \tan\dfrac{x}{2}$ or $-\cot\dfrac{x}{2}$.

70. Prove that $\sin 2\theta = 2\sin\theta\cos\theta$ from the factor expressions of $\sin\theta$ and $\cos\theta$.

71. Resolve $\operatorname{vers}\theta$ into factors without making use of expression for $\sin\theta$.

72. $$\tan^{-1}\left(\frac{\tan 2\theta+\tanh 2\phi}{\tan 2\theta-\tanh 2\phi}\right)+\tan^{-1}\left(\frac{\tan\theta-\tanh\phi}{\tan\theta+\tanh\phi}\right) = \tan^{-1}(\cot\theta\coth\phi).$$

73. $$\sin\theta+\cos\theta = \left(1+\frac{4\theta}{\pi}\right)\left(1-\frac{4\theta}{3\pi}\right)\left(1+\frac{4\theta}{5\pi}\right)\left(1-\frac{4\theta}{7\pi}\right)\ldots$$

and hence

$$\frac{\pi^3}{32} = \frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\ldots$$

74. $$\frac{\pi^2}{64}\left(1-\frac{\pi^2}{12}\right) = \frac{1}{3^4}+\frac{3}{5^4}+\frac{6}{7^4}+\frac{10}{9^4}+\ldots$$

75. $\dfrac{\sinh x}{x} = \cosh \tfrac{1}{2}x \cosh \tfrac{1}{4}x \cosh \tfrac{1}{8}x \ldots$

and hence $\dfrac{1}{x^2} = \operatorname{cosech}^2 x + \dfrac{1}{2^2}\operatorname{sech}^2 \dfrac{x}{2} + \dfrac{1}{4^2}\operatorname{sech}^2 \dfrac{x}{4} + \ldots$

and $\dfrac{1}{\log x} = \dfrac{1}{x-1} + \dfrac{1}{2} \cdot \dfrac{1}{1+x^{\frac{1}{2}}} + \dfrac{1}{4} \cdot \dfrac{1}{1+x^{\frac{1}{4}}} + \ldots$

76. $\dfrac{\pi}{4x} \tanh \dfrac{\pi x}{2} = \dfrac{1}{x^2+1^2} + \dfrac{1}{x^2+3^2} + \dfrac{1}{x^2+5^2} + \ldots$

77. $\dfrac{\pi}{2x} \coth \pi x = \dfrac{1}{2x^2} + \dfrac{1}{x^2+1^2} + \dfrac{1}{x^2+2^2} + \dfrac{1}{x^2+3^2} + \ldots$

78. $\left(1 + \dfrac{2}{1+1^2} + \dfrac{2}{1+2^2} + \dfrac{2}{1+3^2} + \ldots\right)$

$$\left(\frac{1}{4+1^2} + \frac{1}{4+3^2} + \frac{1}{4+5^2} + \ldots\right) = \frac{\pi^2}{8}.$$

79. $\tanh x = \dfrac{x}{1+}\ \dfrac{x^2}{3+}\ \dfrac{x^2}{5+}\ \dfrac{x^2}{7+} \cdots$

80. $\sin x = \dfrac{x}{1+}\ \dfrac{x^2}{6-x^2+}\ \dfrac{6x^2}{20-x^2+}\ \dfrac{20x^2}{42-x^2+} \cdots$

81. $\cos x = \dfrac{1}{1+}\ \dfrac{x^2}{2-x^2+}\ \dfrac{2x^2}{12-x^2+}\ \dfrac{12x^2}{30-x^2+} \cdots$

82. Find the general form of the component in 80 and 81.

83. $\tan^{-1} x = \dfrac{x}{1+}\ \dfrac{x^2}{3-x^2+}\ \dfrac{3^2x^2}{5-3x^2+}\ \dfrac{5^2x^2}{7-5x^2+} \cdots$

84. Calculate π to 3, 4, 5, 6, 7 decimal places respectively from the formulæ 1, 2, 3, 4, 5 of Art. 539.

85. Calculate π to 5 decimal places from the formulæ of Art. 533.

86. The equation $\theta = \cos\theta$ has one and only one root: and this root lies between $\frac{1}{5}\pi$ and $\frac{1}{4}\pi$.

CHAPTER XX.

THE CONSTRUCTION OF TABLES AND INTERPOLATION.

§ 1. The Construction of Tables.

550. *To calculate the value of* π.

$\frac{4}{5}$	= ·8	÷ 1 = · 8	+
× ·04	= ·032	÷ 3 = · 0106666666666667	−
„	= ·00128	÷ 5 = · 256	+
„	= ·0000512	÷ 7 = · 73142857143	−
„	= ·000002048	÷ 9 = · 2275555556	+
„	= ·00000008192	÷ 11 = · 74472727	−
„	= ·0000000032768	÷ 13 = · 2520615	+
„	= ·000000000131072	÷ 15 = · 87381	−
„	= ·0000000000052429	÷ 17 = · 3084	+
„	= ·0000000000002097	÷ 19 = · 110	−
„	= ·0000000000000084	÷ 21 = · 4	+
$1 \div 70^2$	= ·0002040816326531	$\frac{1}{70}$ = · 0142857142857143	−
÷ 70	= ·0000029154518950	÷ 3 = · 9718172983	+
÷ 70	= ·0000000416493128		
÷ 70	= ·0000000005949902	÷ 5 = · 1189980	−
÷ 70	= ·0000000000084998		
÷ 70	= ·0000000000001214	÷ 7 = · 174	+
$1 \div 99^2$	= ·00,01,02,03,04,05,06,07	$\frac{1}{99}$ = · 0101010101010101	+
÷ 99	= ·00,00,01,03,06,10,15,21	÷ 3 = · 3435367174	−
÷ 99	= ·00,00,00,01,04,10,20,35		
÷ 99	= ·00,00,00,00,01,05,15,35	÷ 5 = · 210307	+
÷ 99	= ·00,00,00,00,00,01,06,21		
÷ 99	= ·00,00,00,00,00,00,01,07	÷ 7 = · 15	−
		= ·8103582097472824	+
		·0249600463498340	−
		$\frac{1}{4}\pi$ = ·7853981633974484	
		∴ π = 3·141592653589793	

We will use the formula $\frac{1}{4}\pi = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$.

The multiplier $\frac{1}{25}$, which occurs in $4\tan^{-1}\frac{1}{5}$, may be written $4 \,.\, 10^{-2}$.

Any pair of digits in any power of $\frac{1}{99}$ may be found by adding the preceding pair of digits to the corresponding pair in the preceding power. See above.

551. *To find sin* 10″.

We have $\pi \div 100 = \cdot 03141592653589793,$

$\therefore\ \pi \div 600 = \cdot 00523598775598299,$

$\pi \div 600 \div 9 = \cdot 00058177641733144,$

$\pi \div 600 \div 9 \div 12 = \cdot 00004848136811095.$

Now let θ = circular measure of $10'' = \frac{1}{2}\pi \div 90 \div 60 \div 6$ = above calculated decimal.

Then, by Art. 485, $\sin\theta$ lies between θ and $\theta - \frac{1}{4}\theta^3$ for *any* acute angle.

But if θ measures 10″, $\theta < \cdot 00005$, i.e. $< \frac{1}{2} \cdot 10^{-4}$,

$$\therefore\ \tfrac{1}{4}\theta^3 < \tfrac{1}{32} \cdot 10^{-12} < \tfrac{1}{2} \cdot 10^{-13}.$$

Hence writing θ for $\sin\theta$, the error in sin 10″ will be less than $\frac{1}{2} \cdot 10^{-13}$. That is,

For 13 *places of decimals,* sin 10″ = *circular measure of* 10″ = ·0000484813681.

552. Similarly we have sin 5″ = ·00002424 nearly by halving the circular measure of 10″.

Thus $\sin^2 5'' = (2424 \times 10^{-8})^2 = 5876 \times 10^{-13},$

$$\therefore\ \tfrac{1}{2}(1 - \cos 10'') = \cdot 0000000005876,$$

$$\therefore\ 2(1 - \cos 10'') = \cdot 0000000023504.$$

553. *To calculate the sines of angles which are multiples of* 10″.

If α denotes any angle, we have

$$\sin(n+1)\alpha + \sin(n-1)\alpha = 2\sin n\alpha \cos\alpha,$$

$$\therefore\ \sin(n+1)\alpha - \sin n\alpha = \sin n\alpha - \sin(n-1)\alpha - (2 - 2\cos\alpha)\sin n\alpha.$$

In this equation let $\alpha = 10''$. Then we have found $\sin\alpha$ and $2 - 2\cos\alpha$.

Putting $n = 1$ gives us sin 20″ − sin 10″.

Putting $n = 2$ gives us sin 30″ − sin 20″.

And so on.

The advantage of the above mode of working is that the labour is reduced to the mere multiplication by the small quantity $2 - 2\cos\alpha$.

554. *To test the error involved in the process of calculating the sines of multiples of* 10″.

Whatever error was originally involved in the value used for sin 10″ is increased n. fold in calculating sin n. 10″.

For the error arising from the term $(2 - 2\cos a)\sin na$ is less than $10^{-8} \times$ (the error in sin na) and may, therefore, be neglected.

Assuming then that, if the error in sin a is δ, the errors up to sin na are multiples of δ, we have

$$\sin(n+1)a = 2\sin na - \sin(n-1)a - \ldots$$

in which the error is clearly $2n\delta - (n-1)\delta$, i.e. $(n+1)\delta$.

Thus, by induction, *we have, universally, the error in sin* na is n *times the error in a.*

Let us, then, use the value of sin 10″ which is correct to within $\frac{1}{2}.10^{-13}$. Then the error in 60° would be less than $60 \times 60 \times 6 \times \frac{1}{2}.10^{-13}$, i.e. less than $\frac{1}{2}.10^{-8}$. Hence the above value will enable us to calculate the sines up to 60° correctly to the first 8 decimal places.

555. When the sines of angles up to 60° have been calculated, the remaining sines may be found by simple addition from the formula

(1) $\sin(60° + A) - \sin(60° - A) = 2\cos 60° \sin A = \sin A.$

Or, stopping short at 45°, the remaining sines may be found by approximating to the value of $\sqrt{2}$ from the formula

(2) $\sin(45° + A) - \sin(45° - A) = 2\cos 45° \sin A = \sqrt{2}.\sin A.$

Or, stopping short at 30°, the remaining sines up to 60° may be found by approximating to the value of $\sqrt{3}$ from the formula

(3) $\sin(30° + A) - \sin(30° - A) = 2\cos 30° \sin A = \sqrt{3}.\sin A.$

After this we may use formula (1) to find the remaining sines.

556. Having found the sines, the other ratios may be found from the following:—

$$\cos A = \sin(90° - A),$$

$\tan A = \sin A \div \cos A$ *and* $\tan(45° + A) - \tan(45° - A) = 2\tan 2A$.

$$\cot A = \tan(90° - A),$$
$$\operatorname{cosec} A = \tfrac{1}{2}(\tan \tfrac{1}{2}A + \cot \tfrac{1}{2}A),$$
$$\sec A = \operatorname{cosec}(90° - A).$$

557. The accuracy with which the calculations are performed may be tested by

(1) The known values of the ratios of certain angles (as proved in Chap. IV.), the surd expressions in which may be evaluated.

(2) The known values of the ratios of *certain differences* in the angles.

Thus since the ratios of 3° are known from those of 18° and 15°, the ratios of angles differing by 3° may be successively found independently.

(3) The following two so-called *Formulæ of Verification*, viz.:

$$\sin(36° + A) - \sin(36° - A) - \sin(72° + A) + \sin(72° - A) = \sin A.$$
$$\cos(36° + A) + \cos(36° - A) - \cos(72° + A) - \cos(72° - A) = \cos A.$$

558. The method of constructing the tables of the *logarithms* of given numbers has been explained in Chapter XVIII. Having found the *ratios* of angles as above, we may, therefore, use the table of logarithms to find the *logarithms of the ratios*. But, by the method explained in the following article, we may find the logarithms of the ratios without using the table of the *ratios*.

559. *To find the logarithm of the cosine of any angle.*

In order to calculate logarithms we require to use *factor* formulæ. Thus we have

$$\cos\theta = \left(1 - \frac{2^2\theta^2}{\pi^2}\right)\left(1 - \frac{2^2\theta^2}{3^2\pi^2}\right)\left(1 - \frac{2^2\theta^2}{5^2\pi^2}\right)\cdots$$

Now put $\theta = \frac{m}{n} \cdot \frac{\pi}{2}$. Thus

$$\cos\left(\frac{m}{n}90°\right) = \left(1 - \frac{m^2}{n^2}\right)\left(1 - \frac{m^2}{3^2n^2}\right)\left(1 - \frac{m^2}{5^2n^2}\right)\cdots$$

Taking logarithms, we have $\log \cos \left(\frac{m}{n} 90^\circ\right)$

$= \log (n+m) + \log (n-m) - 2 \log n$

$+ \log (3n+m) + \log (3n-m) - 2 \log 3n + \ldots$

The logarithms of the first one or two factors may thus be found from the table of logarithms; those of the remainder should be found from the formula

$$\log_{10} (1-x) = -\log_{10} e \, (x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \ldots).$$

Thus after finding the logarithms of the first two factors by the tables, we have for the remainder to subtract

$$\log_{10} e \left(\frac{1}{5^2} + \frac{1}{7^2} + \ldots\right) \frac{m^2}{n^2} + \tfrac{1}{2} \log_{10} e \left(\frac{1}{5^4} + \frac{1}{7^4} + \ldots\right) \frac{m^4}{n^4} + \ldots$$

These series converge with sufficient rapidity for purposes of calculation.

560. After calculating the logarithms of any *one* ratio, to find the others requires no more complex operation than *mere subtraction.*

§ 2. The Theory of Interpolation.

561. We shall now examine the basis and limits of the theory of Proportional Differences, which is applied to interpolate values of quantities intermediate between two tabulated values. The general nature of the discussion may be thus indicated.

Let $f(x)$ be some function of x, whose values are tabulated *to an assigned degree of accuracy* for successive values of x *differing by an assigned amount.* We shall show that to a certain determinable degree of accuracy

$$f(x+d) - f(x) = d \times (\text{some function of } x)$$

when d does not exceed a certain amount.

The coefficient of d in the above equation is independent of d, but dependent upon x and upon the form of the function f. It is

called the *Differential Coefficient* of $f(x)$ with respect to x; and is written $f_1(x)$. Thus *approximately*

$$f(x+d) - f(x) = d \,.\, f_1(x).$$

The degree of accuracy of the above equation increases as d decreases. The general object of the discussion is to determine with what degree of accuracy it holds in any case.

562. After showing how far the equation

$$f(x+d) - f(x) = d \,.\, f_1(x)$$

holds true in any case, we may, by varying d and fixing x, write

$$f(x+d') - f(x) = d' \,.\, f_1(x).$$

Thus
$$\frac{f(x+d) - f(x)}{f(x+d') - f(x)} = \frac{d}{d'}.$$

If then d' is the amount by which the successive values of x in the tables differ, this equation gives us either

(1) $f(x+d) - f(x)$ in terms of d,

or (2) d in terms of $f(x+d) - f(x)$.

Though these two problems are theoretically identical, yet they require a different discussion when applying the theory to interpolate in the tables. For, *firstly*, the tabulated differences in x are *constant* so that those in $f(x)$ are *variable;* and, *secondly*, the accuracy with which $f(x)$ is determined for any value of x is *constant*, but that with which x is determined for any value of $f(x)$ is *variable.* In consequence of this distinction, it will be found that $f(x+d)$ must be expanded in powers of d, not $x+d$ in powers of $f(x+d) - f(x)$.

563. *To find the difference of the* logarithms *of two numbers approximately in terms of the difference of the numbers.*

Let d be less than n. Then

$$\log(n+d) - \log n = \log \frac{n+d}{n} = \log\left(1 + \frac{d}{n}\right)$$
$$= \mu\left(\frac{d}{n} - \tfrac{1}{2}\frac{d^2}{n^2} + \tfrac{1}{3}\frac{d^3}{n^3} - \dots\right),$$

where μ stands for the modulus $\log_{10} e$.

In this series the terms are continually decreasing and alternate in sign.

Hence, *the series differs from* $\frac{\mu d}{n}$ *by a quantity less than* $\frac{\mu d^2}{2n^2}$.

Now $\mu = \log_{10} e < \log_9 3$, i.e. $< \frac{1}{2}$.

Suppose then that $\frac{d}{n}$ is not greater than 10^{-p}; then $\frac{d^2}{n^2}$ is not greater than 10^{-2p}.

$$\therefore \frac{\mu d^2}{2n^2} < \tfrac{1}{4} \cdot 10^{-2p}.$$

Hence, if $\frac{d}{n}$ is not greater than 10^{-p}, the equation

$$\log (n + d) - \log n = \mu \frac{d}{n}$$

holds as far as $2p$ decimal places.

564. *To apply the logarithmic-difference formula to the interpolation of the required logarithm of a given number.*

Tables are published of the logarithms of all *integral* numbers from 10000 to 100000. Hence, if we require the logarithm of a fraction lying between two such integers, the ratio of d to n is never greater than 10^{-4}. Hence, if the logarithms were tabulated to 8 decimal places, we might interpolate in such a table, *correctly to* 8 *decimal places*, the logarithms of any fractional numbers by means of the formula

$$\frac{\log (n + d) - \log n}{\log (n + 1) - \log n} = \frac{d}{1}.$$

565. Conversely, *to apply the logarithmic-difference formula to the interpolation of the required number corresponding to a given logarithm.*

The logarithms are tabulated to 7 decimal places. In the difference formula, put $d = 1$. Thus

$$\log (n + 1) - \log n = \frac{\mu}{n} = N \cdot 10^{-7} \text{ say.}$$

Here N is the tabulated difference in the logarithms corresponding to the difference 1 in the numbers at any part n of the table. Hence we can find N different numbers between $n+1$ and n whose logarithms differ in the first 7 places of decimals. Hence, at this part of the table, we can find accurately to an N^{th} of unity any intermediate number whose logarithm is given correctly to 7 decimal places.

Now N varies inversely as n.

Hence, as we proceed in the table, we can interpolate, between numbers differing by the constant amount unity, a continually smaller number of values corresponding to given logarithms calculated to a fixed number 7 of decimal places.

566. *To find the difference of the* sines *of two angles approximately in terms of the difference of the angles.*

Let $\delta < 1$ be the circular measure of an angle. Then

$$\begin{aligned}\sin(\alpha+\delta) - \sin\alpha &= \cos\alpha \sin\delta - \sin\alpha\,(1-\cos\delta)\\ &= \cos\alpha\,(\delta - \tfrac{1}{6}\delta^3 + \ldots) - \sin\alpha\,(\tfrac{1}{2}\delta^2 - \tfrac{1}{24}\delta^4 + \ldots).\end{aligned}$$

In this series after $\delta\cos\alpha$ the terms, taken in pairs, are continually decreasing and alternate in sign.

Hence the series differs from $\delta\cos\alpha$ by a quantity less than $\frac{1}{2}\delta^2\sin\alpha + \frac{1}{6}\delta^3\cos\alpha$.

Let δ be the circular measure of an angle less than $1'$. Then

$$\delta < \frac{3{\cdot}15}{180\times 60} < \frac{{\cdot}0027}{9} < {\cdot}0003 < 3\,.\,10^{-4},$$

$$\therefore\ \delta^2 < 9\,.\,10^{-8} \text{ and } \delta^3 < 27\,.\,10^{-12}.$$

Also $\sin\alpha$ and $\cos\alpha$ are < 1.

$$\therefore\ \tfrac{1}{2}\delta^2\sin\alpha + \tfrac{1}{6}\delta^3\cos\alpha < \tfrac{1}{2}\,.\,10^{-7}.$$

Hence, if δ is not greater than the circular measure of $1'$, the equation

$$\sin(\alpha+\delta) - \sin\alpha = \delta\cos\alpha$$

holds as far as 7 decimal places.

567. *To interpolate the* sine *of any given angle between* a *and* $a+1'$, we may use the above formula which will give the required sine correctly to 7 decimal places.

568. But conversely, *to interpolate an* angle *corresponding to a given sine*, in the sine-difference formula put for δ the circular measure of $1'$. Then

$$\sin(a+1') - \sin a = \frac{\pi}{10800}\cos a = N.10^{-7} \text{ say.}$$

Here N is the tabulated difference in the sines corresponding to the difference $1'$ in the angles at any part a of the table. Hence we can find N different angles between $a+1'$ and a whose sines differ in the first 7 decimal places.

Now as the angle a increases up to 90°, $\cos a$ and therefore N decreases. Hence we can find intermediate angles from their sines with continually decreasing accuracy as we approach 90°.

For example, $\sin 78°\ 5'$ and $\sin 78°\ 6'$ differ by 600.10^{-7}. Hence up to $78°\ 6'$ we can, from the sines of angles given correctly to 7 places, interpolate angles correctly to within a 600^{th} of a minute; i.e. a 10^{th} of a second. Again $\sin 84°\ 3'$ and $\sin 84°\ 4'$ differ by 300.10^{-7}. Hence up to this point we can similarly interpolate correctly to within a 5^{th} of a second. And so on.

569. *To find the difference of the* tangents *of two angles approximately in terms of the difference of the angles.*

Here we have

$$\tan(a+\delta) - \tan a = \frac{\sin(a+\delta)\cos a - \cos(a+\delta)\sin a}{\cos(a+\delta)\cos a}$$

$$= \frac{\sin\delta}{\cos^2 a\cos\delta - \cos a\sin a\sin\delta} = \frac{\sec^2 a}{\cot\delta - \tan a}.$$

Now $\cot\delta < \dfrac{1}{\delta}$. Hence the difference of the tangents

$$> \frac{\sec^2 a}{1/\delta - \tan a}, \text{ à } \textit{fortiori} > \delta\sec^2 a.$$

But $\cot \delta > \frac{1}{\delta} - \frac{\delta}{2}$. Hence the difference of the tangents

$$< \frac{\delta \sec^2 a}{1 - \delta \tan a - \frac{1}{2}\delta^2}.$$

Thus $\tan(a+\delta) - \tan a$ differs from $\delta \sec^2 a$ by a quantity less than

$$\frac{\delta^2 \sec^2 a \,(\tan a + \frac{1}{2}\delta)}{1 - \delta \tan a - \frac{1}{2}\delta^2} = \textit{nearly } \delta^2 \sec^2 a \tan a.$$

This quantity increases as a and as δ increase.

Now it will be found by examining the tables that approximately $\tan a \sec^2 a = 1$, when a is 34°; $= 10$, when a is 64°; $= 100$, when a is 78°; $= 1000$, when a is 84°: and so increases more and more rapidly up to 90°, when it is infinite. Hence the tangent-difference formula is less and less accurate as the angle approaches 90°.

570. Hence, *to interpolate* a tangent *corresponding to a given angle between* a *and* $a + 1'$, the equation

$$\tan(a+\delta) - \tan a = \delta \sec^2 a$$

holds for 7 places, up to 34°; for 6 places, up to 64°; for 5 places, up to 78°; for 4 places, up to 84°; and so on.

571. Conversely, *to interpolate an* angle *corresponding to a given tangent*, in the tangent-difference formula put for δ the circular measure of 1′. Thus

$$\tan(a+1') - \tan a = \frac{\pi}{10800} \sec^2 a = N \,.\, 10^{-7} \text{ say.}$$

Here N, which varies as $\sec^2 a$, increases as a increases. Its smallest value, viz. the difference between tan 0 and tan 1′, is about 3000. Hence even here we could find an angle correctly to within nearly $\frac{1}{10}$ of a second corresponding to a tangent given correctly to 7 decimal places.

Now the difference in the tangents for 1′ is at 34° about 4000×10^{-7}; at 64°, 1500×10^{-6}; at 78°, 600×10^{-5}; at 84°, 270×10^{-4}; and so on.

Hence we can find angles, when the tangents are given correctly to 7 decimal places from 0 to 34°, with an accuracy increasing from $\frac{1}{50}$ to $\frac{1}{70}$ of a second: when the tangents are given to 6 places from 34° to 64°, with an accuracy increasing from $\frac{1}{7}$ to $\frac{1}{25}$ of a second: when the tangents are given to 5 places from 64° to 78°, with an accuracy increasing from $\frac{1}{2}$ to $\frac{1}{10}$ of a second: when the tangents are given to 4 places from 78° to 84°, with an accuracy increasing from 1 to $\frac{1}{4}$ of a second: and so on.

572. *To find the difference of the* secants *of two angles approximately in terms of the difference of the angles.*

Here we have

$$\sec(\alpha+\delta)-\sec\alpha=\frac{\sec\alpha\sec\delta}{1-\tan\alpha\tan\delta}-\sec\alpha$$

$$=\sec\alpha\sec\delta(1+\tan\alpha\tan\delta+\tan^2\alpha\tan^2\delta+\ldots)-\sec\alpha.$$

Now $\sec\delta>1$ and $\tan\delta>\delta$. Hence the difference of the secants $>\delta\sec\alpha\tan\alpha$.

But $\cos\delta>1-\frac{1}{2}\delta^2$ and $\sin\delta<\delta$. Hence the difference of the secants

$$<\frac{\sec\alpha}{1-\delta\tan\alpha-\frac{1}{2}\delta^2}-\sec\alpha.$$

Hence $\sec(\alpha+\delta)-\sec\alpha$ differs from $\delta\sec\alpha\tan\alpha$ by a quantity

$$<\frac{\delta^2\sec\alpha(\frac{1}{2}+\tan^2\alpha+\frac{1}{2}\delta\tan\alpha)}{1-\delta\tan\alpha-\frac{1}{2}\delta^2}=\textit{nearly}\ \delta^2\sec\alpha(\sec^2\alpha-\tfrac{1}{2}).$$

This quantity increases as α and as δ increase.

Now it will be found by examining the tables that approximately $\sec\alpha(\sec^2\alpha-\frac{1}{2})=1$ when α is 33°; $=10$ when α is 63°; $=100$ when α is 77°; $=1000$ when α is 84°; and so increases more and more rapidly up to 90° where it is infinite.

Hence the secant-difference formula is less and less accurate as we approach 90°.

573. Hence, *to interpolate a* secant *corresponding to a given angle between a and* $a + 1'$, the equation

$$\sec(a+\delta) - \sec a = \delta \sec a \tan a$$

holds for 7 places up to 33°; for 6 places up to 63°; for 5 places up to 77°; for 4 places up to 84°; and so on.

574. Conversely, *to interpolate an* angle *corresponding to a given secant*, in the secant-difference formula put for δ the circular measure of 1′. Thus

$$\sec(a+1') - \sec a = \frac{\pi}{10800} \sec a \tan a = N \,.\, 10^{-7} \text{ say.}$$

Here N increases as a increases. Hence, as we approach *zero*, we can interpolate an angle with less and less accuracy corresponding to a given secant. For instance sec 11° 28′ and sec 11° 29′ differ by 600×10^{-7}. Hence here we can find angles to within $\frac{1}{10}$ of a second corresponding to secants given correctly to 7 places. Again sec 3° 13′ and sec 3° 14′ differ by 160×10^{-7}. Hence here we can interpolate an angle correctly to within only $\frac{3}{8}$ of a second. And so on.

On the other hand as we approach 90° we find the following results. The difference in the secants for 1′ is at first about 2×10^{-7}, and at 33° about 2000×10^{-7}; at 63°, 1200×10^{-6}; at 77°, 600×10^{-5}; at 84°, 250×10^{-4}; and so on.

Hence we can find angles, when the secants are given correctly to 7 decimal places from 1′ to 33°, with an accuracy increasing from 30″ to $\frac{1}{30}$ of a second; when the secants are given correctly to 6 decimal places from 33° to 63°, with an accuracy increasing from $\frac{1}{3}$ to $\frac{1}{20}$ of a second; when the secants are given correctly to 5 decimal places from 63° to 77°, with an accuracy increasing from $\frac{1}{2}$ to $\frac{1}{10}$ of a second; when the secants are given correctly to 4 decimal places from 77° to 84°, with an accuracy increasing from 1 to $\frac{1}{4}$ of a second; and so on.

575. The secondary ratios cosine, cotangent, and cosecant of an angle being simply the sine, tangent, and secant respectively

of the complementary angle need not be treated separately. As the secondary ratios vary incongruently with the angle, their differences will be negative when the difference in the angle is positive. Thus approximately

$$\cos\ (\alpha+\delta)-\cos\ \alpha=-\delta\sin\alpha,$$
$$\cot\ (\alpha+\delta)-\cot\ \alpha=-\delta\operatorname{cosec}^2\alpha,$$
$$\operatorname{cosec}\ (\alpha+\delta)-\operatorname{cosec}\alpha=-\delta\operatorname{cosec}\alpha\cot\alpha.$$

Precisely similar limits of accuracy attach to these equations as to those of the primary ratios; if we substitute 0 for 90°, greater angle for less angle, and so on.

576. *To find the difference in the* log sines *of two angles in terms of the difference of the angles.*

Here we have

$$\log\sin(\alpha+\delta)-\log\sin\alpha=\log(\cos\delta+\cot\alpha\sin\delta).$$

Now $\cos\delta<1$ and $\sin\delta<\delta$.

Hence the difference of the log sines is $<\log(1+\delta\cot\alpha)$

$$<\mu(\delta\cot\alpha-\tfrac{1}{2}\delta^2\cot^2\alpha+\ldots)\text{ if }\delta\cot\alpha<1.$$

This series has terms alternate in sign and decreasing in magnitude. Hence, *à fortiori*, the difference in log sines is $<\mu\delta\cot\alpha$.

Again

$$\log\sin(\alpha+\delta)-\log\sin\alpha=\log\cos\delta+\log(1+\cot\alpha\tan\delta).$$

Now $\cos\delta>1-\tfrac{1}{2}\delta^2$ and $\tan\delta>\delta$.

Hence the difference of the log sines is

$$>\mu\{\delta\cot\alpha-\tfrac{1}{2}\delta^2(1+\cot^2\alpha)+\tfrac{1}{3}\delta^3\cot^3\alpha-\tfrac{1}{4}\delta^4(\tfrac{1}{2}+\cot^4\alpha)+\ldots\}.$$

Thus $\log\sin(\alpha+\delta)-\log\sin\alpha$ differs from $\mu\delta\cot\alpha$ by a quantity which is $<\frac{1}{2}\mu\delta^2\operatorname{cosec}^2\alpha$.

This quantity decreases as α increases.

Now if δ measures an angle not greater than 1′, $\frac{1}{2}\mu\delta^2$ is not greater than 2.10^{-8}.

Now it will be found on examining the tables that approximately $\operatorname{cosec}^2\alpha=5$ when α is 27°; $=50$ when α is 8°; $=500$

when a is $2\frac{1}{3}°$; $= 5000$ when a is $48'$: and so increases more and more rapidly down to 0 where it is infinite.

Hence the log sine difference formula is less and less accurate as we approach 0.

577. Hence, *to interpolate a* log sine *corresponding to a given angle between* a *and* $a + 1'$, the equation

$$\log \sin (a + \delta) - \log \sin a = \mu\delta \cot a$$

holds for 7 places down to 27°; for 6 places down to 8°; for 5 places down to $2\frac{1}{3}°$; for 4 places down to $48'$; and so on.

578. Conversely, *to interpolate an* angle *corresponding to a given log sine*, in the log sine difference formula put for δ the circular measure of $1'$. Thus

$$\log \sin (a + 1') - \log \sin a = \frac{\mu\pi}{10800} \cot a = N \,.\, 10^{-7} \text{ say.}$$

Here N varies as cot a and therefore increases as a decreases. Hence, as we approach zero, we find the difference in the log sines for $1'$ is at 27° about 2478×10^{-7}; at 8°, 900×10^{-6}; at $2\frac{1}{3}°$, 300×10^{-5}; at $48'$, 90×10^{-4}.

Hence we can find angles, when the log sines are given correctly to 7 decimal places down to 27°, with an accuracy that increases up to $\frac{1}{40}$ of a second; when the log sines are given correctly to 6 places from 27° down to 8°, with an accuracy that increases from $\frac{1}{4}$ to $\frac{1}{15}$ of a second; when the log sines are given correctly to 5 places from 8° down to $2\frac{1}{3}°$, with an accuracy that increases from $\frac{2}{3}$ to $\frac{1}{5}$ of a second; when the log sines are given correctly to 4 places from $2\frac{1}{3}°$ to $48'$, with an accuracy that increases from 2 to $\frac{2}{3}$ of a second: and so on.

As we approach 90°, cot a decreases; hence, we can interpolate angles from their log sines with continually less and less accuracy as the angle increases to 90°. Thus

The difference in log sines for $1'$ is at 64° $35'$ about 600×10^{-7}; at 81° $54'$ about 180×10^{-7}; at 84° $34'$ about 120×10^{-7}; at 87° $19'$ about 60×10^{-7}; and so on.

Thus when the log sines are given correctly to 7 decimal places, we may find the corresponding angle at 64° 35′ within $\frac{1}{10}$ of a second; at 81° 54′ within $\frac{1}{3}$ of a second; at 84° 34′ within $\frac{1}{2}$ a second; at 87° 19′ within 1 second; and so on.

579. Since approximately

$$\log \sin (\alpha + \delta) - \log \sin \alpha = \mu\delta \cot \alpha \quad \ldots\ldots\ldots\ldots\ldots\text{I.}$$

$\therefore$ changing α into $\frac{1}{2}\pi + \alpha$,

$$\log \cos (\alpha + \delta) - \log \cos \alpha = -\mu\delta \tan \alpha \ldots\ldots\ldots\ldots\text{II.}$$

$\therefore$ subtracting II. from I,

$$\log \tan (\alpha + \delta) - \log \tan \alpha = 2\mu\delta \operatorname{cosec} 2\alpha \ldots\ldots\ldots \text{III.}$$

Again, since $\log (1 \div x) = -\log x$,

from III, $\log \cot (\alpha + \delta) - \log \cot \alpha = -2\mu\delta \operatorname{cosec} 2\alpha \ldots\ldots\text{IV.}$

from II, $\log \sec (\alpha + \delta) - \log \sec \alpha = \mu\delta \tan \alpha \ldots\ldots\ldots\ldots\text{V.}$

from I, $\log \operatorname{cosec} (\alpha + \delta) - \log \operatorname{cosec} \alpha = -\mu\delta \cot \alpha \ldots\ldots\text{VI.}$

Similar limits of accuracy attach to these latter equations as to I.

580. We have seen that as the angle decreases to zero, the degree of accuracy with which we can interpolate from the log sine difference formula decreases rapidly. Hence it is important to overcome this difficulty. The method is explained in the following articles.

581. We have $\log \sin \theta = \log \dfrac{\sin \theta}{\theta} + \log \theta \quad \ldots\ldots\ldots\ldots\ldots\text{I.}$

$$\text{Now } \log \frac{\sin (\theta + \delta)}{\theta + \delta} - \log \frac{\sin \theta}{\theta} = \log \frac{\sin (\theta + \delta)}{\sin \theta} - \log \frac{\theta + \delta}{\theta}$$

$$= \log (1 + \delta \cot \theta - \tfrac{1}{2}\delta^2 + \ldots) - \log \left(1 + \frac{\delta}{\theta}\right)$$

$$= \mu\delta \left(\cot \theta - \frac{1}{\theta}\right) - \tfrac{1}{2}\mu\delta^2 \left(\operatorname{cosec}^2 \theta - \frac{1}{\theta^2}\right) + \ldots$$

The terms in this series after the first are alternate in sign and diminishing in magnitude. Hence the difference of the

functions $\log \frac{\sin\theta}{\theta}$ differs from $\mu\delta\left(\cot\theta - \frac{1}{\theta}\right)$ by a quantity less than

$$\tfrac{1}{2}\mu\delta^2\left(\operatorname{cosec}^2\theta - \frac{1}{\theta^2}\right).$$

Now when θ is small $\operatorname{cosec}^2\theta$ is nearly equal to $\frac{1}{\theta^2}$; hence, δ being less than the circular measure of 1′, the above quantity is less than 10^{-7}.

Hence a table of functions of the form $\log(\sin\theta \div \theta)$ for every minute would be one in which we could use the formula of proportional differences correctly to 7 decimal places.

It is necessary to explain how such a table could be used.

582. *To explain how a table of functions of the form* $\log(\sin\theta \div \theta)$ *could be used.*

Let θ contain n seconds. Then equation I. of last article may be written

$$L\sin n'' = \log\frac{\sin\theta}{\theta} + 10 + \log\sigma + \log n \ldots\ldots\ldots \text{II.}$$

where σ denotes the circular measure of $1'' = 4{\cdot}848137 \times 10^{-6}$.

By reference to the table of logarithms we shall find that $\log\sigma = \bar{6}{\cdot}6855749$.

Hence a table might be constructed in which the constant 4·6855749 is added to the value of $\log(\sin\theta \div \theta)$ for every minute (or smaller interval) of small angles.

Then to find the log sine corresponding to an angle of n'', we should add to this tabulated value the logarithm of n found from the table of logarithms. Moreover since n would not be more than 100000 we could interpolate for fractional values of n if required to find $\log n$.

Conversely, to find the angle corresponding to a log sine, we should first interpolate in the ordinary tables to find an *approximate* value of the angle. We should then use this value to find

$\log \frac{\sin\theta}{\theta} + 10 + \log\sigma$ from the tables: and subtract this from the given $L\sin$ in order to find $\log n$ by equation II. The error arising from using this approximation would be proportional to $\mu(\cot\theta - 1/\theta)$, which is a very small quantity compared with $\mu\cot\theta$, to which the error in $\log\sin\theta$ is proportional.

The above method is known as *Delambre's Method.*

583. A modification of Delambre's Method is applicable to the log sines and also to the log tangents of small angles. It is called *Maskelyne's Method.*

We have

$$L\sin n'' = \log\frac{\sin\theta}{\theta} + 10 + \log\sigma + \log n \ldots\ldots\ldots \text{II.}$$

$$L\tan n'' = \log\frac{\tan\theta}{\theta} + 10 + \log\sigma + \log n \ldots\ldots\ldots \text{III.}$$

Now $\sin\theta = \theta - \frac{1}{6}\theta^3$ *and* $\tan\theta = \theta + \frac{1}{3}\theta^3$ *and* $\cos\theta = 1 - \frac{1}{2}\theta^2$ approximately for small angles.

$$\therefore \frac{\sin\theta}{\theta} = (\cos\theta)^{\frac{1}{3}} \text{ and } \frac{\tan\theta}{\theta} = (\cos\theta)^{-\frac{2}{3}} \text{ approximately.}$$

$$\therefore \log\frac{\sin\theta}{\theta} = \tfrac{1}{3}\log\cos\theta \text{ and } \log\frac{\tan\theta}{\theta} = -\tfrac{2}{3}\log\cos\theta.$$

Hence the equations II. and III. may be used by employing the ordinary tables of the log cosines.

584. The method of argument in this section may be summed up as follows:

By expanding $f(x+d) - f(x)$ in powers of d, we find that it is approximately equal to a quantity involving d. It differs, however, from this quantity approximately by a second quantity involving d^2.

The limit of the term involving d^2 assigns the limit of accuracy with which the formula of proportional differences may be used.

Thus, in interpolating a value of $f(x)$ when the value of x is given, the degree of accuracy *varies inversely* with this *second* term which involves d^2.

But, in interpolating a value of x when the value of $f(x)$ is given, the degree of accuracy *varies directly* with the first term which involves d.

585. When the term involving d^2 is so *large* as to inconveniently restrict the degree of accuracy with which a value of $f(x)$ may be interpolated, then the differences in $f(x)$ are said to be **irregular.**

When the term involving d is so *small* as to inconveniently restrict the degree of accuracy with which a value of x may be interpolated, then the differences in $f(x)$ are said to be **insensible.**

586. In the preceding two articles it is essential to note that the possibility of applying the formula of proportional differences depends primarily upon the smallness of the second term involving d^2.

Hence the *absolute* smallness of this term measures the accuracy for interpolating $f(x)$ from x.

And the *relative* smallness of this term as compared with the first term measures the accuracy for interpolating x from $f(x)$.

587. The general conclusions are summed up by using the terms *irregular* and *insensible.*

I. The differences in $\sin a$ are *insensible* as a approaches $90°$: hence the accuracy for interpolating a from $\sin a$ diminishes as a increases.

II. The differences in $\tan a$ and in $\sec a$ are *irregular* as a approaches $90°$; hence the accuracy for interpolating $\tan a$ or $\sec a$ from a diminishes as a increases: and the differences in $\tan a$ and $\sec a$ are *insensible* as a approaches zero; hence the accuracy for interpolating a from $\tan a$ or $\sec a$ diminishes as a diminishes.

III. The differences in $\log \sin \alpha$ and $\log \tan \alpha$ are *irregular* as α approaches zero; hence the accuracy for interpolating $\log \sin \alpha$ or $\log \tan \alpha$ diminishes as α diminishes: and the differences in $\log \sin \alpha$ and $\log \tan \alpha$ are *insensible* as α approaches $90°$; hence the accuracy for interpolating α from $\log \sin \alpha$ or $\log \tan \alpha$ diminishes as α increases.

Errors in measurement.

588. The determination of a required length or angle by means of trigonometrical formulæ illustrates the value of the investigations of this chapter.

Thus the required magnitude is a certain function of magnitudes *directly* measured. But any magnitude *directly measured* is subject to an *error*, whose limit is generally known. Hence it becomes important to evaluate the possible error in the *derived* magnitude.

A few examples will illustrate this point.

Example 1. The height h of a tower is determined by measuring a horizontal line a from its base and the angle of elevation θ of the top of the tower at the end of the line. Find the possible error in the height h due to the possible error δ in the angle θ.

Here we have $h = a \tan \theta$ for estimated height

and $h' = a \tan (\theta + \delta)$ for real height.

$\therefore$ possible error, i.e. $h' - h = a \{\tan (\theta + \delta) - \tan \theta\}$

$= a\delta \sec^2 \theta$ approximately.

Hence the *absolute* possible error varies as $\sec^2 \theta$, and therefore increases as θ increases.

But the *relative* possible error, i.e. the ratio of the absolute possible error to the whole height $= \dfrac{a\delta \sec^2 \theta}{a \tan \theta} = 2\delta \operatorname{cosec} 2\theta$: which is least when $\theta = 45°$.

Example 2. The height h of a hill is determined by measuring the angles of elevation α and β of the top and bottom of a tower of height b on the top of the hill. Find the possible error in h due to a possible error δ in the angle α.

Here we have $(h+b)\cot a = h\cot\beta$,

$$\therefore h = \frac{b\cot a}{\cot\beta - \cot a} = \frac{b}{\tan a\cot\beta - 1},$$

$$\text{and } h' = \frac{b}{\tan(a+\delta)\cot\beta - 1},$$

$$\therefore \frac{h-h'}{h'} = \frac{\tan(a+\delta)\cot\beta - 1}{\tan a\cot\beta - 1} - 1$$

$$= \frac{\delta\sec^2 a\cot\beta}{\tan a\cot\beta - 1} \text{ approximately,}$$

$$= \delta\{\cot(a-\beta) + \tan a\},$$

which gives the relative error.

EXAMPLES XX.

1. Show that the equation $\sin\theta = \theta$ may be used with the following degrees of accuracy:

From zero to 18′ up to 7 places: from zero to 52′ up to 6 places; from zero to 95′ up to 5 places; from zero to 164′ up to 4 places.

Hence calculate the sines of 7′ 12″; 30′ 15″; 1°; and 2° 11′ 2″ as accurately as is possible from the formula $\sin\theta = \theta$.

2. Given that the moon subtends at the earth an angle of half a degree, find the distance at which a circular plate of six inches diameter must be placed so as just to conceal it.

3. Given that the radius of the earth is 3960 miles, show that the number of *miles* in the visible horizon from a point of observation at a given number of *feet* above the ground may be found, approximately, by adding to the number of feet its half and taking the square root of the result.

4. Given that the tangents of 32° 11′, 32° 12′, 57° 48′, 57° 49′ are respectively ·6293274, ·6297336, 1·5879731, 1·5889979, find the cosecant of 64° 22′ 26″.

5. Given that $\sin(45° + A) = \cdot 71$, show that A is, approximately, 14′ or 89° 46′, the circular measure of 1′ being ·0002909.

6. Given that $\dfrac{\sin\theta}{\theta} = \dfrac{1733}{1734}$, show that θ is the circular measure of 3° 22′ approximately.

7. Given that the tangents of 21° 20′ and 34° 20′ are ·3905541 and ·6830066 respectively, find the tangent of 55° 40′.

8. From a table of tangents calculated for every minute, show that an angle may be determined to within about $\frac{1}{60}$th part of a second when the angle is about 30°: and to within about $\frac{1}{20}$th part of a second when the angle is about 60°.

9. From a table of log sines calculated for every minute, show that an angle may be determined to within about $\frac{2}{3}$ of a second when the angle is about 88° or about $3\frac{1}{2}°$.

10. Find $L\sin 1° 40′ 15″$ by interpolating in the table of log sines directly. Then find its value by Maskelyne's method: and hence compare the degrees of accuracy of the two methods.

11. Find the angle whose log sine is $\bar{2}\cdot 7523456$.

12. If the angle C of a triangle be very obtuse, the number of seconds in its defect from 180° is very nearly

$$206265\sqrt{\left\{\frac{2(a+b)(a+b-c)}{ab}\right\}}.$$

13. An angle $\alpha + \delta$ is found by interpolating in the table of logarithmic sines given for every minute.

Show that the circular measure of the error involved is, approximately,

$$(\mu\delta - \delta^2)\operatorname{cosec} 2\alpha,$$

where μ is the circular measure of 1′.

Hence show that the maximum error, in calculating an angle between α and $\alpha + 1′$, is about $0″\cdot 00435 \operatorname{cosec} 2\alpha$.

14. If the area of a triangle is calculated by measuring two sides and the included angle as 197 ft., 2030 ft., and 60° respectively, and if there is an error of 2″ in the measurement of the angle, show that this will involve an error of about 1 sq. ft. in the area.

15. If the angles B and C of a triangle are found from A, b, c: and if there is a small error δ in A; show that the errors in B and C are to one another as $\tan B$ to $\tan C$: and hence that they are, respectively,

$$-\delta \sin B \cos C \operatorname{cosec} A \quad \textit{and} \quad -\delta \sin C \cos B \operatorname{cosec} A.$$

16. If errors a', b', c' are made in the measurement of the three sides a, b, c of a triangle, show that the error in A is

$$-\tfrac{1}{2}a\,(b' \cos C + c' \cos B - a')/S.$$

17. If an error a' is made in a and no errors are made in b and c; then r', r_1', R', the errors involved in r, r_1, R respectively, will be given by

$$r' = a'\,(R/s - \tfrac{1}{2}r_1);\ r_1' = a'(R/s - \tfrac{1}{2}r);$$
$$R' = a' \cot B \cot C \operatorname{cosec} A.$$

18. The height h of an object is estimated by measuring its angles of elevation α, β, γ at three points A, B, C in a straight line, such that $BC = a$, $AC = b$, $AB = c$. If α', β', γ' be the errors in α, β, γ, show that the error in h is

$$\frac{h^3}{abc}\left(\frac{a\cos\alpha}{\sin^3\alpha}\cdot\alpha' - \frac{b\cos\beta}{\sin^3\beta}\cdot\beta' + \frac{c\cos\gamma}{\sin^3\gamma}\cdot\gamma'\right).$$

CHAPTER XXI.

IMAGINARY AND COMPLEX QUANTITIES.

§ 1. General Theorems.

589. It was pointed out, in the introduction to Chap. XI., that the square of every *positive or negative* quantity is *positive*. Hence, a quantity whose square is *negative*, is neither *positive nor negative*. Such a quantity is usually called *Imaginary;* while positive or negative quantities are called *Real*.

In this chapter we shall exhibit the use of *Imaginary* quantities. By their means the results of the preceding ten chapters may be systematised and extended. We shall not *interpret* imaginary quantities at present, but shall merely use them *symbolically:* i.e. we shall deduce results by their use which involve only *real* quantities. To accomplish this it is necessary to postulate that

Imaginary quantities shall be governed by the same laws of operation as real quantities.

In the same way (as was indicated in the introduction to Chap. XI.) negative quantities may be introduced symbolically without interpretation, by assuming that they follow the same laws as positive quantities. It was afterwards shown that the relation of negative to positive could be used to represent *oppositeness of direction*.

590. In confining ourselves to *positive* and *negative* quantities, we confine ourselves to quantities which have the *same* or the *opposite* sign: i.e. the addition of two such quantities is always equivalent to addition or subtraction.

So in introducing *imaginary* quantities, these may be either positive imaginary or negative imaginary. That is, we shall imply that the addition to any imaginary of a *negative* imaginary is equivalent to the subtraction of the corresponding positive.

Thus we have *four* affections of quantity; viz. positive real, negative real, positive imaginary, and negative imaginary.

591. If a is any positive real quantity, $\sqrt{(-a)}$ is an imaginary quantity, to which we may attribute either sign.

But since we postulate that all our symbols shall obey the same laws, we have (see Art. 233)

$$\sqrt{(-a)} = \sqrt{\{a \times (-1)\}} = \sqrt{a} \times \sqrt{(-1)}.$$

Hence *the square root of any negative number can be expressed as the product of a real number into the square root of* -1.

Thus the only symbol we need use is $\sqrt{(-1)}$.

The symbol i will be used as the equivalent of the *positive* value of the $\sqrt{(-1)}$; and $-i$, therefore, as the *negative* value of the $\sqrt{(-1)}$.

592. *Wholly imaginary quantities.* A real number $\times i$ is called a *wholly imaginary* quantity.

Wholly imaginary quantities are incomparable with wholly real quantities.

Thus, if a and b are both real quantities, the equation $a = bi$ involves a contradiction, unless $a = 0$ and $b = 0$.

Thus, the one equation

$$a = bi$$

is interpreted to mean the same as the *two* equations

$$a = 0 \text{ and } b = 0.$$

This is the fundamental principle which enables us to make use of symbols of imaginary quantities.

593. *Complex quantities.* The sum of a wholly real and a wholly imaginary quantity is called a *complex* quantity.

Such a sum cannot be interpreted at present; but its use is explained by a corollary from the last article. Thus

If a, b, c, d are wholly real, the equation $a+bi=c+di$ is equivalent to $a-c=(d-b)i$.

Hence it is also equivalent to the *two* equations

$$a-c=0 \text{ and } d-b=0, \text{ i.e. } a=c \text{ and } b=d.$$

Thus if one complex expression is equal to another, *we may equate separately the real and imaginary terms.*

In the symbol $a+bi$, we may suppose that either a or b is zero: so that complex quantities include wholly imaginary and wholly real quantities. The remainder of this chapter deals mainly with complex quantities.

Conjugates, Norms, and Moduli.

594. The complex quantities $y+zi$ and $y-zi$ are called *Conjugates* of one another.

The quantity y^2+z^2 is called the *Norm* of $y+zi$.

The *positive* value of the *square root* of the Norm is called the *Modulus.*

Thus the *norm* of *either* of two conjugate complex quantities is their *product.*

And the *modulus* of *either* of two conjugate complex quantities is their *geometric mean.*

595. *The necessary and sufficient condition that a complex quantity should* vanish *is that its* norm [*or* modulus] *should vanish.*

For, if y and z are real and not zero, y^2 and z^2 must both be *positive.* Hence y^2+z^2 cannot vanish unless both y and z vanish. Thus, if $y^2+z^2=0$, $y+zi=0$.

Conversely, if $y+zi=0$, we must have $y=0$ and $z=0$; and $\therefore\ y^2+z^2=0$.

596. *The* norm [*or* modulus] *of the* product *of two complex quantities is equal to the* product *of their* norms [*or* moduli].

For $$(a+bi)(c+di) = ac - bd + (ad+bc)i,$$

and $$(ac-bd)^2 + (ad+bc)^2 = a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2$$
$$= (a^2+b^2)(c^2+d^2).$$

Cor. The norm [or modulus] of the n^{th} power of a complex quantity is the n^{th} power of its norm [or modulus].

597. *If the* product *of two complex quantities* vanishes *one of the* factors *must vanish.*

For, by Art. 595, in order that $(a+bi)(c+di)$ may vanish, its *norm* must vanish.

That is, by Art. 596, $(a^2+b^2)(c^2+d^2)$ must vanish.

$\therefore$ either a^2+b^2 or c^2+d^2 must vanish.

$\therefore$ by Art. 595, either $a+bi$ or $c+di$ must vanish.

Theorems on Equations.

598. The theorems on *factors* in Arts. 412, 413, 414 may be expressed as theorems on *equations*, and thus be applied to *any complex* (including real) quantities.

Thus, if a_1, a_2, a_3... are (complex) roots of the equation $f(x)=0$ (a_1, a_2, a_3 being all different), then $f(a_1)=0$, $f(a_2)=0$, $f(a_3)=0$, and so on. Thus, for all values of x,

$$f(x) \equiv \phi(x)(x-a_1)(x-a_2)(x-a_3)\ldots\ldots$$

If $f(x)$ is of the n^{th} degree, px^n being its highest term, and if $a_1, a_2 \ldots a_n$ are n roots of $f(x)=0$; then, for all values of x,

$$f(x) \equiv p(x-a_1)(x-a_2)\ldots(x-a_n).$$

599. Hence, *an equation of the* n$^{\text{th}}$ *degree cannot have more than* n *roots.*

For, if possible, let a_{n+1} be an $(n+1^{\text{th}})$ root of the above equation $f(x)=0$: so that $f(a_{n+1})=0$.

Then, putting $x=a_{n+1}$ in the above identity,

$$f(a_{n+1}) = p(a_{n+1}-a_1)(a_{n+1}-a_2)\ldots(a_{n+1}-a_n).$$

But the factor p is not zero, because, by hypothesis, the equation is of the n^{th} degree. Hence one of the factors $a_{n+1}-a_1$, $a_{n+1}-a_2$,... must be zero by Art. 597. That is, a_{n+1} must be equal to one of the roots $a_1, a_2 \ldots a_n$.

600. *The complex roots of an equation with* real coefficients *enter in* conjugate pairs.

Let $f(x)$ be any integral function of x, in which the coefficients of the several powers of x are all *real*.

Then writing $y+zi$ for x, the real terms must be those which involve *even* powers, and the imaginary terms those which involve *odd* powers of z. Hence

If $f(y+zi) \equiv P+Qi$, then $f(y-zi) \equiv P-Qi$.

Therefore, if $y+zi$ is a root of $f(x)=0$; we have $P=0$ and $Q=0$; and $\therefore f(y-zi)=0$. That is

If $y+zi$ is a root of $f(x)=0$, then $y-zi$ is also a root of $f(x)=0$.

601. *To express the product of any number of factors of the form* $\cos\alpha + \mathrm{i}\sin\alpha$ *as a complex quantity.*

We have shown geometrically, that

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

and

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Hence, since $i^2=-1$, $(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)$

$$= \cos\alpha\cos\beta + i\sin\alpha\cos\beta + i\cos\alpha\sin\beta - \sin\alpha\sin\beta$$
$$= \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

Hence again $(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma)$

$$= \{\cos(\alpha+\beta)+i\sin(\alpha+\beta)\}(\cos\gamma+i\sin\gamma)$$
$$= \cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma).$$

And so on. Thus the product of any number of factors

$$(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)\ldots(\cos\xi+i\sin\xi)$$
$$= \cos(\alpha+\beta+\ldots+\xi)+i\sin(\alpha+\beta+\ldots+\xi).$$

602. *To find the cosine and sine of the* sum *of any number of angles in terms of the cosines and sines of the angles.*

Multiply out the product on the left-hand side of the above equation: and equate the real and imaginary parts. Then; observing that

$$i^2=-1\,;\ i^3=-i\,;\ i^4=+1\,;\ i^5=i\,;\ \text{and so on}\,;$$

we have

$$\cos(\alpha+\beta+\ldots+\xi)=\cos\alpha\cos\beta\ldots\cos\xi-\Sigma\sin\alpha\sin\beta\cos\gamma\ldots\cos\xi$$
$$+\Sigma\sin\alpha\sin\beta\sin\gamma\sin\delta\cos\epsilon\ldots\cos\xi-\ldots$$
$$\sin(\alpha+\beta+\ldots+\xi)=\Sigma\sin\alpha\cos\beta\ldots\cos\xi-\Sigma\sin\alpha\sin\beta\sin\gamma\cos\delta\ldots\cos\xi$$
$$+\ldots$$

where the symbol Σ denotes 'the sum of all terms of the form obtained by interchanging the letters α, β, $\gamma\ldots\xi$ in the product to which it is attached.'

This conclusion is the same as that obtained in Art. 350. The two methods of proof should be carefully compared.

De Moivre's Theorem.

603. As in Art. 601 we have, for all real values of m and n,

$$(\cos m\theta+i\sin m\theta)(\cos n\theta+i\sin n\theta)=\cos(m+n)\theta+i\sin(m+n)\theta.$$

That is, calling $\cos m\theta+i\sin m\theta$, $f(m)$,

$$f(m)\times f(n)=f(m+n).$$

Hence, by the Index Theorem of Art. 463, m and n being real,

$$f(n)\text{ is one value of }\{f(1)\}^n,$$

i.e. $\cos n\theta+i\sin n\theta$ is one value of $(\cos\theta+i\sin\theta)^n$,

for all positive and negative values of n.

This result is so important that we will give a proof of it without introducing the index theorem in its general form.

It is usually known as *De Moivre's theorem.*

604. *Whatever positive or negative value* n *may have,*

$\cos n\theta + i \sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

We have $\cos(\theta+\phi+\psi+\ldots) + i\sin(\theta+\phi+\psi+\ldots)$

$= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)(\cos\psi + i\sin\psi)\ldots$.

Now let $\theta = \phi = \psi = \ldots$ and let there be n of these angles. Thus, if n is a positive integer,

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n \quad \ldots\ldots\ldots\ldots (1).$$

Next let n be a negative integer $= -m$ say. Then

$$\begin{aligned}\cos n\theta + i\sin n\theta &= \cos m\theta - i\sin m\theta \\ &= \frac{\cos^2 m\theta - i^2\sin^2 m\theta}{\cos m\theta + i\sin m\theta} \quad \ldots \\ &= \frac{1}{(\cos\theta + i\sin\theta)^m} \text{ by (1)} \\ &= (\cos\theta + i\sin\theta)^{-m} \quad \ldots\ldots\ldots\ldots (2).\end{aligned}$$

Lastly, let n be a fraction $= \frac{p}{q}$ say, where p and q are integral.

Then $(\cos n\theta + i\sin n\theta)^q = \left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)^q$

$= \cos p\theta + i\sin p\theta$ by (1) and (2)

$= (\cos\theta + i\sin\theta)^p$ by (1) and (2)

$\therefore \cos n\theta + i\sin n\theta$ is one of the values of $\sqrt[q]{(\cos\theta + i\sin\theta)^p}$: or

$\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

605. *To find* q *values of* $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$ *where* p *and* q *are prime to one another.*

We have $\cos\theta + i\sin\theta = \cos(\theta + 2\lambda\pi) + i\sin(\theta + 2\lambda\pi)$ where λ has any integral value.

$\therefore$ *any one* of the values of $\cos\frac{p}{q}(\theta + 2\lambda\pi) + i\sin\frac{p}{q}(\theta + 2\lambda\pi)$ is a value of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$.

Giving to λ any two values μ and μ', the expressions

$$\cos\frac{p}{q}(\theta + 2\mu\pi) + i\sin\frac{p}{q}(\theta + 2\mu\pi)$$

and $$\cos\frac{p}{q}(\theta+2\mu'\pi)+i\sin\frac{p}{q}(\theta+2\mu'\pi)$$

would be equal, only if $\cos\frac{p}{q}(\theta+2\mu\pi)=\cos\frac{p}{q}(\theta+2\mu'\pi)$

and $$\sin\frac{p}{q}(\theta+2\mu\pi)=\sin\frac{p}{q}(\theta+2\mu'\pi);$$

i.e. if the angles $\frac{p}{q}(\theta+2\mu\pi)$ and $\frac{p}{q}(\theta+2\mu'\pi)$ were equi-cosinal and equi-sinal;

i.e. if $\frac{p}{q}(\theta+2\mu\pi)\sim\frac{p}{q}(\theta+2\mu'\pi)$ were a multiple of 2π;

i.e. if $p(\mu\sim\mu')$ were a multiple of q.

If then p and q are prime to one another, this condition becomes 'if $\mu\sim\mu'$ is a multiple of q.'

Hence, if we give to λ the q values $0, 1, 2\ldots n-1$, we shall obtain q different values of $\cos\frac{p}{q}(\theta+2\lambda\pi)+i\sin\frac{p}{q}(\theta+2\lambda\pi)$: and, if we give to λ higher values, the values of this expression will simply recur.

Hence q values of $(\cos\theta+i\sin\theta)^{\frac{p}{q}}$ are given by the q values of $\cos\frac{p}{q}(\theta+2\lambda\pi)+i\sin\frac{p}{q}(\theta+2\lambda\pi)$.

606. *To find* q *values of the* q$^{\text{th}}$ *root of any complex quantity.*

Let $a+bi$ be any complex quantity: and let ρ denote the *positive* value of $\sqrt{(a^2+b^2)}$. Then ρ is the **modulus** of $a+bi$.

Since a/ρ and b/ρ are each numerically less than unity, and are such that *the sum of their squares is equal to unity*, therefore, there is one and only one angle between 0 and 2π whose cosine is a/ρ and whose sine is b/ρ. Let this angle be θ.

Thus $$a+bi=\rho(\cos\theta+i\sin\theta).$$

Now, since ρ is positive, there cannot be *more than one* positive q^{th} root of ρ. And there *is one* positive q^{th} root of ρ,

whose value can be approximately determined by the Binomial Theorem.

Let $\rho^{\frac{1}{q}}$ denote this *positive value* of $\sqrt[q]{\rho}$.

Then, by the proposition of the last article, q values of the q^{th} root of $\cos\theta + i\sin\theta$ are given by the q values of

$$\cos\frac{\theta+2\lambda\pi}{q} + i\sin\frac{\theta+2\lambda\pi}{q}.$$

Hence q values of the q^{th} root of $a+bi$ are given by putting λ successively equal to 0, 1, 2,...$q-1$ in the expression

$$\rho^{\frac{1}{q}}\left(\cos\frac{\theta+2\lambda\pi}{q} + i\sin\frac{\theta+2\lambda\pi}{q}\right).$$

607. There is one point in the above result that requires special attention.

In the theory of indices, we use the symbol $a^{\frac{p}{q}}$ as the equivalent of $\sqrt[q]{(a^p)}$ or $(\sqrt[q]{a})^p$.

Now, by the theory of fractions, $a^{\frac{p}{q}} = a^{\frac{rp}{rq}}$. Is it then legitimate to infer that $\sqrt[q]{(a^p)} = \sqrt[rq]{(a^{rp})}$?

By the above proposition $\sqrt[q]{(a^p)}$ has q values; and, by Art. 599, it cannot have more than q values. Similarly $\sqrt[rq]{(a^{rp})}$ has rq values and no more.

Hence $\sqrt[q]{(a^p)}$ and $\sqrt[rq]{(a^{rp})}$ have not the same number of values: and, therefore, cannot be said to have the same meaning.

The q values of $\sqrt[q]{(a^p)}$ are, however, all contained amongst the rq values of $\sqrt[rq]{(a^{rp})}$: as the student may show for himself.

The roots of $+1, -1, +\mathrm{i}$, *and* $-\mathrm{i}$.

608. In De Moivre's theorem, Art. 605, let $p=1$.

Now put $\theta=0$. Thus $\cos\theta=1$, and $\sin\theta=0$.

Hence the q q^{th} roots of *unity* are given by the q values of

$$\cos\frac{2\lambda\pi}{q} + i\sin\frac{2\lambda\pi}{q}.$$

Next put $\theta = \pi$. Thus $\cos\theta = -1$ and $\sin\theta = 0$.

Hence the q q^{th} roots of *minus unity* are given by the q values of

$$\cos\frac{2\lambda+1}{q}\pi + i\sin\frac{2\lambda+1}{q}\pi.$$

Next put $\theta = \frac{1}{2}\pi$. Thus $\cos\theta = 0$ and $\sin\theta = 1$.

Hence the q q^{th} roots of i are given by the q values of

$$\cos\frac{4\lambda+1}{q}\cdot\frac{\pi}{2} + i\sin\frac{4\lambda+1}{q}\cdot\frac{\pi}{2}.$$

Next put $\theta = \frac{3}{2}\pi$. Thus $\cos\theta = 0$ and $\sin\theta = -1$.

Hence the q q^{th} roots of $-i$ are given by the q values of

$$\cos\frac{4\lambda+3}{q}\cdot\frac{\pi}{2} + i\sin\frac{4\lambda+3}{q}\cdot\frac{\pi}{2}.$$

609. Again, in Art. 606, put $\theta = 0$. Then $a + bi$ reduces to a, i.e. ρ, a *real positive* quantity.

Hence the q^{th} roots of a are

$$a^{\frac{1}{q}}\left(\cos\frac{2\lambda\pi}{q} + i\sin\frac{2\lambda\pi}{q}\right),$$

where the expression in the bracket represents the q^{th} roots of *unity*.

Again put $\theta = \pi$. Then $a + bi$ reduces to a, i.e. $-\rho$, a *real negative* quantity.

Hence the q^{th} roots of a are

$$(-a)^{\frac{1}{q}}\left(\cos\frac{2\lambda+1}{q}\pi + i\sin\frac{2\lambda+1}{q}\pi\right),$$

where the expression in the bracket represents the q^{th} roots of *minus unity*.

We may illustrate the important result of Art. 606 by showing that the ratio of any two q^{th} roots of a quantity is a q^{th} root of unity.

Thus giving to λ any two values μ and μ', the ratio of the two corresponding q^{th} roots of $a+bi$ is

$$\frac{\cos(\theta+2\mu\pi)/q + i\sin(\theta+2\mu\pi)/q}{\cos(\theta+2\mu'\pi)/q + i\sin(\theta+2\mu'\pi)/q}.$$

either $(\theta + 2\mu\pi/n) + (\theta + 2\mu'\pi/n)$ *or* $(\theta + 2\mu\pi/n) \sim (\theta + 2\mu'\pi/n)$ is a multiple of 2π;

i.e. unless either $n\theta$ is a multiple of π, or $\mu \sim \mu'$ is a multiple of n.

If $n\theta$ is a multiple of π, $\cos n\theta = \pm 1$: which has been excluded.

Hence, giving to λ the n values 0, 1, 2...$n-1$ we have $2n$ values of x.

Hence

$$x^{2n} - 2x^n \cos n\theta + 1 \equiv P_{\lambda=0}^{\lambda=n-1} \left\{x - \cos\left(\theta + \frac{2\lambda\pi}{n}\right) \mp i \sin\left(\theta + \frac{2\lambda\pi}{n}\right)\right\}$$

$$= P_{\lambda=0}^{\lambda=n-1} \left\{x^2 - 2x\cos\left(\theta + \frac{2\lambda\pi}{n}\right) + 1\right\},$$

where the two factors, obtained by taking the upper and lower signs before i, are multiplied together.

613. Put $\cos\theta + i\sin\theta = x$.

Multiply by $\cos\theta - i\sin\theta$. Thus

$$1 = x(\cos\theta - i\sin\theta) \quad \text{or} \quad \cos\theta - i\sin\theta = x^{-1}.$$

Also, by De Moivre's theorem, n being integral

$$\cos n\theta + i\sin n\theta = x^n \quad \text{and} \quad \cos n\theta - i\sin n\theta = x^{-n}.$$

Moreover by addition and subtraction

$$2\cos\theta = x + x^{-1}; \quad 2i\sin\theta = x - x^{-1}.$$

$$2\cos n\theta = x^n + x^{-n}; \quad 2i\sin n\theta = x^n - x^{-n}.$$

These are extremely useful transformations.

We showed, in Arts. 416, 418, that

$$x^n + x^{-n} \text{ and } (x^n - x^{-n}) \div (x - x^{-1})$$

are respectively the same functions of $x + x^{-1}$ as $2\cos n\theta$ and $2\sin n\theta/2\sin\theta$ are of $2\cos\theta$. But we did not there *equate* the corresponding expressions. To do this we must implicitly introduce imaginaries. For $2\cos\theta$ must be < 2 for any *real* angle, and $x + x^{-1}$ must be > 2 for any *real* value of x ($\because x - 2 + x^{-1}$ is a square quantity).

614. In Art. 610, where n is even, write $2m$ for n and divide both sides by x^m. Thus

$$x^m - x^{-m} = (x - x^{-1}) \times$$

$$\left(x + x^{-1} - 2\cos\frac{\pi}{m}\right)\left(x + x^{-1} - 2\cos\frac{2\pi}{m}\right)\ldots\left(x + x^{-1} - 2\cos\frac{m-1}{m}\pi\right).$$

Now put $x + x^{-1} = 2\cos\theta$; then $x - x^{-1} = 2i\sin\theta$; and

$$x^m - x^{-m} = 2i\sin m\theta.$$

Thus $$\sin m\theta = \sin\theta \times$$

$$\left(2\cos\theta - 2\cos\frac{\pi}{m}\right)\left(2\cos\theta - 2\cos\frac{2\pi}{m}\right)\ldots\left(2\cos\theta - 2\cos\frac{m-1}{m}\pi\right).$$

Again, in Art. 611, where n is even, write $2m$ for n, and divide both sides by x^m. Thus

$$x^m + x^{-m} =$$

$$\left(x + x^{-1} - 2\cos\frac{\pi}{2m}\right)\left(x + x^{-1} - 2\cos\frac{3\pi}{2m}\right)\ldots\left(x + x^{-1} - 2\cos\frac{2m-1}{2m}\pi\right).$$

Now put $x + x^{-1} = 2\cos\theta$; then $x^m + x^{-m} = 2\cos m\theta$. Thus

$$2\cos m\theta =$$

$$\left(2\cos\theta - 2\cos\frac{\pi}{2m}\right)\left(2\cos\theta - 2\cos\frac{3\pi}{2m}\right)\ldots\left(2\cos\theta - 2\cos\frac{2m-1}{2m}\pi\right).$$

Again in Art. 612, divide both sides by x^n. Thus

$$+ x^{-n} - 2\cos n\theta = (x + x^{-1} - 2\cos\theta)\left\{x + x^{-1} - 2\cos\left(\theta + \frac{2\pi}{n}\right)\right\}\ldots$$

$$\left\{x + x^{-1} - 2\cos\left(\theta + \frac{n-1}{n}2\pi\right)\right\}.$$

Now put $x + x^{-1} = 2\cos\alpha$; then $x^n + x^{-n} = 2\cos n\alpha$. Thus

$$2\cos n\alpha - 2\cos n\theta = (2\cos\alpha - 2\cos\theta)\left\{2\cos\alpha - 2\cos\left(\theta + \frac{2\pi}{n}\right)\right\}\ldots$$

$$\left\{2\cos\alpha - 2\cos\left(\theta + \frac{n-1}{n}2\pi\right)\right\}.$$

These 6 results are the same as those obtained in Arts. 419 420, 421.

§ 2. Expansions by De Moivre's Theorem.

615. In the following propositions we shall require to make use of the **Principle of Continuity.**

Def. A function of a variable, whose value may be made to change by as small a quantity as we please, by making the change in the variable sufficiently small, is said to be **continuous.**

Prop. An endless series in ascending integral powers of x is a continuous function of x, for all values for which it is convergent.

For let, $f(x) \equiv c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$

Then changing x into $x + d$, we have

$$f(x+d) \equiv c_0 + c_1 (x+d) + c_2 (x+d)^2 + \ldots + c_n (x+d)^n + \ldots$$

Here the first term in $c_n (x+d)^n$ is $c_n x^n$.

$\therefore f(x+d) - f(x) \equiv d \,.\, S$ where S is finite, since $f(x+d)$ is convergent.

Now, if a is any assignable quantity *not zero*, we can make d less than $a \div S$, since S is *not infinite*. Hence $f(x+d) - f(x)$ may be made less than assignable quantity a by sufficiently diminishing d. That is, $f(x)$ is *continuous*.

The above proposition is required, when we know the value of a series for some *particular* value of x, but cannot otherwise determine which of a number of different forms must be assigned to its *general* value. The proposition shows that we must assign such a form to the *general* value of the series as will make it change *continuously* from its known value when x changes continuously.

616. In deducing expansions involving complex quantities, our results are based on the Binomial Theorem.

We have, then, *first* to examine whether the expansions are *convergent*. This will require that both the series of *real* terms and the series of *imaginary* terms are convergent.

We have, *secondly*, to show how the proof of the Binomial Theorem is to be applied to *complex quantities*. The student will notice, then, that the *index theorem* required that the index m involved should be *real* (positive or negative) and that $f(m)$ should be *single-valued*. If these conditions are fulfilled $f(m)$ may be *itself* complex. Then, if $f(m) \times f(n) = f(m+n)$, $f(n)$ is *one* of the values of $f(1)^n$. This theorem applies, as shown in Arts. 465, 466, to any series of the form of the Binomial Expansion, whether x be complex or real, *provided* n *is real*. It will remain, therefore, to determine to which of the values of $f(1)^n$ the series must be equated, when n is fractional.

617. In De Moivre's theorem, let n be a positive integer. Then $$\cos n\theta + i \sin n\theta = (\cos\theta + i \sin\theta)^n.$$

Expanding, by the binomial theorem for a positive integer, and equating the real and imaginary terms, we have

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{1.2}\cos^{n-2}\theta \sin^2\theta + \ldots$$

$$\sin n\theta = n\cos^{n-1}\theta \sin\theta - \frac{n(n-1)(n-2)}{1.2.3}\cos^{n-3}\theta \sin^3\theta + \ldots$$

Dividing we have

$$\tan n\theta = \frac{n\tan\theta - \frac{1}{6}n(n-1)(n-2)\tan^3\theta + \ldots}{1 - \frac{1}{2}n(n-1)\tan^2\theta + \ldots}$$

$$\cot n\theta = \frac{\cot^n\theta - \frac{1}{2}n(n-1)\cot^{n-2}\theta + \ldots}{n\cot^{n-1}\theta - \frac{1}{6}n(n-1)(n-2)\cot^{n-3}\theta + \ldots}$$

These results have already been obtained (Art. 359).

618. Let n be a negative integer, $= -m$ say.

(1) Let $\tan\theta < 1$. Then

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n = \cos^n\theta\,(1 + i\tan\theta)^n.$$

We may expand in ascending powers of $\tan\theta$, since $\tan\theta$ is < 1. Thus, equating real and imaginary terms,

$$\cos n\theta = \cos^n\theta\{1 - \tfrac{1}{2}n(n-1)\tan^2\theta + \ldots\}$$

$$\sin n\theta = \cos^n\theta\{n\tan\theta - \tfrac{1}{6}n(n-1)(n-2)\tan^3\theta + \ldots\}$$

where n is negative. Or

$$\cos m\theta = \sec^m\theta\{1 - \tfrac{1}{2}m(m+1)\tan^2\theta + \ldots\}$$
$$\sin m\theta = \sec^m\theta\{m\tan\theta - \tfrac{1}{6}m(m+1)(m+2)\tan^3\theta + \ldots\}$$

where m is positive.

(2) Let $\tan\theta > 1$. Then

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n = \sin^n\theta\,(i + \cot\theta)^n.$$

$\therefore\ i^{-n}\cos n\theta$ (n even) or $i^{-n+1}\sin n\theta$ (n odd)
$$= \sin^n\theta\{1 - \tfrac{1}{2}n(n-1)\cot^2\theta + \ldots\}$$

$i^{-n+2}\sin n\theta$ (n even) or $i^{-n+1}\cos n\theta$ (n odd)
$$= \sin^n\theta\{n\cot\theta - \tfrac{1}{6}n(n-1)(n-2)\cot^3\theta + \ldots\}$$

where n is negative. Or

$(-1)^{\frac{m}{2}}\cos m\theta$ (m even) or $(-1)^{\frac{m-1}{2}}\sin m\theta$ (m odd)
$$= \operatorname{cosec}^m\theta\{1 - \tfrac{1}{2}m(m+1)\cot^2\theta + \ldots\}$$

$(-1)^{\frac{m+2}{2}}\sin m\theta$ (m even) or $(-1)^{\frac{m-1}{2}}\cos m\theta$ (m odd)
$$= \operatorname{cosec}^m\theta\{m\cot\theta - \tfrac{1}{6}m(m+1)(m+2)\cot^3\theta + \ldots\}$$

where m is positive.

619. Let n be fractional $= p/q$ say.

Then, by De Moivre's theorem, the q values of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$ are given by $\cos\frac{p}{q}\phi + i\sin\frac{p}{q}\phi$ where ϕ is any angle whose cosine $= \cos\theta$ and whose sine $= \sin\theta$.

Now suppose $\tan\theta < 1$.

Then, by the binomial theorem, since p/q is real so that the argument of Art. 466 holds although $\cos\theta + i\sin\theta$ is complex, the q values of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$ are given by the series

$$\sqrt[q]{(\cos^p\theta)}\left\{1 - \frac{p(p-q)}{\underline{|2}}\left(\frac{\tan\theta}{q}\right)^2 + \frac{p(p-q)(p-2q)(p-3q)}{\underline{|4}}\left(\frac{\tan\theta}{q}\right)^4 - \ldots.\right\}$$

$$+ i\sqrt[q]{(\cos^p\theta)}\left\{p\frac{\tan\theta}{q} - \frac{p(p-q)(p-2q)}{\underline{|3}}\left(\frac{\tan\theta}{q}\right)^3 + \ldots.\right\}$$

where, by Art. 606, the multiplier $\sqrt[q]{(\cos^p\theta)}$ has q values; and the series into which it is multiplied is finite and *single-valued*, since $\tan\theta < 1$.

Now suppose $\cos\theta$ is *positive*.

Then $\sqrt[q]{(\cos^p\theta)}$ has one *real positive value*. Using this value in the above series, the upper line is wholly real and the lower line is wholly imaginary.

Hence the upper line $= \cos\frac{p}{q}\phi$ and the lower line $= i\sin\frac{p}{q}\phi$, where ϕ is some angle whose cosine $= \cos\theta$ and whose tangent $= \tan\theta$.

It remains to determine which of the alternative values of ϕ is to be taken. To do this we apply the *principle of continuity*. Thus

As $\cos\theta$ approaches the limit $+1$ and $\tan\theta$ the limit 0, the upper line approaches the limit $+1$ and the lower line the limit 0. Hence, in the limit, the upper line $= \cos\frac{p}{q}0$ and the lower line $= i\sin\frac{p}{q}0$. Hence the *general* values of the upper and lower line are, respectively, $\cos\frac{p}{q}\theta$ and $i\sin\frac{p}{q}\theta$, where θ is the (positive or negative) *acute* angle whose cosine $= \cos\theta$ and whose tangent $= \tan\theta$.

For, if any other values, e.g. $\cos\frac{p}{q}(2\pi+\theta)$ and $i\sin\frac{p}{q}(2\pi+\theta)$ were given, then as θ approached the limit 0, the two series would *approach the limits* $\cos\frac{p}{q}2\pi$ and $i\sin\frac{p}{q}2\pi$, and yet when θ *reached* its limit, the two series would be 1 and 0 respectively.

This would involve a breach of continuity, which we have shown to be impossible in a convergent series.

Hence we conclude as follows:

If n is fractional, and if $\tan\theta < 1$, and if $\cos\theta$ is positive, then
$\cos^n\theta\{1 - \frac{1}{2}n(n-1)\tan^2\theta + \frac{1}{24}n(n-1)(n-2)(n-3)\tan^4\theta\ldots\} = \cos n\theta$
$\cos^n\theta\{n\tan\theta - \frac{1}{6}n(n-1)(n-2)\tan^3\theta + \ldots\} = \sin n\theta$
where θ is *acute*, and $\cos^n\theta$ has its *real positive* value.

620. Next let $\cos\theta$ be negative, and let q be odd.

Then taking the real negative value of $\sqrt[q]{(\cos^p\theta)}$ if p is odd, and the real positive value of $\sqrt[q]{(\cos^p\theta)}$ if p is even, the upper line is wholly real and the lower line wholly imaginary.

Now as $\cos\theta$ approaches the limit -1 and $\tan\theta$ the limit 0, the upper line approaches the limit $+1$ or -1 according as p is even or odd, and the lower line the limit 0. Hence, in the limit, the upper line $= \cos\frac{p}{q}q\pi$ and the lower line $= i\sin\frac{p}{q}q\pi$. Hence, by the principle of continuity, the general values of the upper and lower line are, respectively, $\cos\frac{p}{q}\theta$ and $i\sin\frac{p}{q}\theta$, where θ is the angle *nearest to* $q\pi$ whose cosine $= \cos\theta$ and whose tangent $= \tan\theta$.

621. Next let $\cos\theta$ be negative; and q even, but $\frac{1}{2}q$ odd.

Then p and q being supposed prime to one another, p is *odd*. Then, by Art. 608,

$$\sqrt[q]{(\cos^p\theta)} = \sqrt[q]{(-\cos^p\theta)} \cdot \left(\cos\frac{2\lambda+1}{q}p\pi + i\sin\frac{2\lambda+1}{q}p\pi\right)$$

where λ has any integral value. Put $2\lambda + 1 = \frac{1}{2}q$. Then one value of $\sqrt[q]{(\cos^p\theta)}$ is $i\sqrt[q]{(-\cos^p\theta)} \cdot (-1)^{\frac{p-1}{2}}$, where $\sqrt[q]{(-\cos^p\theta)}$ has its *real positive* value.

Giving to $\sqrt[q]{(\cos^p\theta)}$ this value, the upper line becomes wholly imaginary and the lower line wholly real.

Thus

$$\sqrt[q]{(-\cos^p\theta)}\left\{1 - \frac{p(p-q)}{\underline{|2}}\left(\frac{\tan\theta}{q}\right)^2 + \ldots\right\} = (-1)^{\frac{p-1}{2}}\sin\frac{p}{q}\phi;$$

$$\sqrt[q]{(-\cos^p\theta)}\left\{p\frac{\tan\theta}{q}-\frac{p(p-q)(p-2q)}{\lfloor 3}\left(\frac{\tan\theta}{q}\right)^3+\ldots.\right\}$$
$$=(-1)^{\frac{p+1}{2}}\cos\frac{p}{q}\phi;$$

where ϕ is *some* angle whose cosine $=\cos\theta$ and whose tan $=\tan\theta$.

Now as $\cos\theta$ approaches the limit -1 and $\tan\theta$ the limit 0, the upper line here approaches the limit 1 and the lower line the limit 0.

Hence, in the limit, upper line $=(-1)^{\frac{p-1}{2}}\sin\frac{p}{q}\cdot\frac{q\pi}{2}$;

and lower line $=(-1)^{\frac{p+1}{2}}\cos\frac{p}{q}\cdot\frac{q\pi}{2}$.

Hence the general values of the upper and lower line are, respectively, $(-1)^{\frac{p-1}{2}}\sin\frac{p}{q}\theta$ and $(-1)^{\frac{p+1}{2}}\cos\frac{p}{q}\theta$, where θ is the angle *nearest to* $\frac{1}{2}q\pi$, whose cosine $=\cos\theta$ and whose tangent $=\tan\theta$.

622. Let $\cos\theta$ be negative, and p and q have any values prime to one another. Then, by Art. 608,

$$\sqrt[q]{(\cos^p\theta)}=\sqrt[q]{(-\cos\theta)^p}\left(\cos\frac{2r+1}{q}p\pi+i\sin\frac{2r+1}{q}p\pi\right).$$

Hence giving to $\sqrt[q]{(-\cos\theta)^p}$ its *real positive* value, we have

$$\cos\frac{2r+1}{q}p\pi\times\sqrt[q]{(-\cos\theta)^p}\left\{1-\frac{p(p-q)}{\lfloor 2}\left(\frac{\tan\theta}{q}\right)^2+\ldots\right\}$$
$$-\sin\frac{2r+1}{q}p\pi\times\sqrt[q]{(-\cos\theta)^p}\left\{p\frac{\tan\theta}{q}-\frac{p(p-q)(p-2q)}{\lfloor 3}\left(\frac{\tan\theta}{q}\right)^3+\ldots\right\}$$
$$=\cos\frac{p}{q}\theta;\text{ and}$$
$$\sin\frac{2r+1}{q}p\pi\times\sqrt[q]{(-\cos\theta)^p}\left\{1-\frac{p(p-q)}{\lfloor 2}\left(\frac{\tan\theta}{q}\right)^2+\ldots\right\}$$
$$+\cos\frac{2r+1}{q}p\pi\times\sqrt[q]{(-\cos\theta)^p}\left\{p\frac{\tan\theta}{q}-\frac{p(p-q)(p-2q)}{\lfloor 3}\left(\frac{\tan\theta}{q}\right)^3+\ldots\right\}$$
$$=\sin\frac{p}{q}\theta,$$

where θ is the angle nearest to $(2r+1)\pi$ whose cosine $=\cos\theta$ and whose tangent $=\tan\theta$.

This result includes those of the preceding two articles.

For, if q is odd, we may put $2r+1=q$, so that $\cos\frac{2r+1}{q}p\pi=(-1)^p$, and $\sin\frac{2r+1}{q}p\pi=0$. Hence the series for $\cos p\theta/q$ and $\sin p\theta/q$ reduce to the forms given in Art. 620.

And, if q is even and $\frac{1}{2}q$ odd, we may put $2r+1=\frac{1}{2}q$, so that $\cos\frac{2r+1}{q}p\pi=0$ and $\sin\frac{2r+1}{q}p\pi=(-1)^{\frac{p-1}{2}}$. Hence the series reduce to the forms given in Art. 621.

Moreover, if q and $\frac{1}{2}q$ are *both even*, we cannot make either $\cos p(2r+1)\pi/q$ or $\sin p(2r+1)\pi/q$ vanish, so that $\cos p\theta/q$ and $\sin p\theta/q$ cannot be expressed in a single series.

623. The result of the last article follows immediately from Art. 619.

For, if $\theta-(2r+1)\pi$ is *acute*, the multipliers of

$$\cos(2r+1)p\pi/q \text{ and } \sin(2r+1)p\pi/q$$

in the series of Art. 622 are, by Art. 619, respectively

$$\cos\{\theta-(2r+1)\pi\}p/q \textit{ and } \sin\{\theta-(2r+1)\pi\}p/q.$$

Hence the expressions given for $\cos p\theta/q$ and $\sin p\theta/q$ are equal respectively to

$$\cos\left(\frac{2r+1}{q}p\pi+\frac{p}{q}\theta-\frac{2r+1}{q}p\pi\right) \textit{and} \sin\left(\frac{2r+1}{q}p\pi+\frac{p}{q}\theta-\frac{2r+1}{q}p\pi\right).$$

624. Lastly suppose $\tan\theta>1$.

Then $$(\cos\theta+i\sin\theta)^{\frac{p}{q}}=(i\sin\theta)^{\frac{p}{q}}(1-i\cot\theta)^{\frac{p}{q}}.$$

If $\sin\theta$ is positive, by Art. 608,

$$(i\sin\theta)^{\frac{p}{q}}=\sqrt[q]{(\sin^p\theta)}\left(\cos\frac{4r+1}{q}\cdot\frac{p\pi}{2}+i\sin\frac{4r+1}{q}\cdot\frac{p\pi}{2}\right).$$

If $\sin\theta$ is negative, by Art. 608,

$$(i\sin\theta)^{\frac{p}{q}}=\sqrt[q]{(-\sin\theta)^p}\left(\cos\frac{4r+3}{q}\cdot\frac{p\pi}{2}+i\sin\frac{4r+3}{q}\cdot\frac{p\pi}{2}\right),$$

where $\sqrt[q]{(\sin^p\theta)}$ and $\sqrt[q]{(-\sin\theta)^p}$ are taken as *real positive*.

Write C for the series $1 - \frac{p(p-q)}{1.2}\left(\frac{\cot\theta}{q}\right)^2 + \ldots$

and S for the series $p\,\frac{\cot\theta}{q} - \frac{p(p-q)(p-2q)}{1.2.3}\cdot\left(\frac{\cot\theta}{q}\right)^3 + \ldots$

Here, when $\cot\theta = 0$, $C = 1$ and $S = 0$. This gives us the clue for discovering their general value.

Thus, if $\sin\theta$ is positive,

$$\sqrt[q]{(\sin^p\theta)}\left(C\,.\cos\frac{4r+1}{q}\,.\,\frac{p\pi}{2} + S\,.\sin\frac{4r+1}{q}\,.\,\frac{p\pi}{2}\right) = \cos\frac{p}{q}\theta,$$

and $$\sqrt[q]{(\sin^p\theta)}\left(C\,.\sin\frac{4r+1}{q}\,.\,\frac{p\pi}{2} - S\,.\cos\frac{4r+1}{q}\,.\,\frac{p\pi}{2}\right) = \sin\frac{p}{q}\theta,$$

where θ is the angle nearest to $\frac{1}{2}(4r+1)\pi$ whose sine $= \sin\theta$ and whose cot $= \cot\theta$.

But, if $\sin\theta$ is negative,

$$\sqrt[q]{(-\sin\theta)^p}\left(C\,.\cos\frac{4r+3}{q}\,.\,\frac{p\pi}{2} + S\,.\sin\frac{4r+3}{q}\,.\,\frac{p\pi}{2}\right) = \cos\frac{p}{q}\theta,$$

and $$\sqrt[q]{(-\sin\theta)^p}\left(C\,.\sin\frac{4r+3}{q}\,.\,\frac{p\pi}{2} - S\,.\cos\frac{4r+3}{q}\,.\,\frac{p\pi}{2}\right) = \sin\frac{p}{q}\theta,$$

where θ is the angle nearest to $\frac{1}{2}(4r+3)\pi$, whose sine $= \sin\theta$ and whose cot $= \cot\theta$. Thus

(A) If p is even, q of the form $4r+1$, and $\sin\theta$ positive,

$$C\,.\sqrt[q]{(\sin^p\theta)} = (-1)^{\frac{p}{2}}\cos\frac{p}{q}\theta \text{ and } S\,.\sqrt[q]{(\sin^p\theta)} = (-1)^{\frac{p+2}{2}}\sin\frac{p}{q}\theta.$$

(B) If p is even, q of the form $4r+3$, and $\sin\theta$ negative,

$$C\,.\sqrt[q]{(-\sin\theta)^p} = (-1)^{\frac{p}{2}}\cos\frac{p}{q}\theta \text{ and } S\,.\sqrt[q]{(-\sin\theta)^p} = (-1)^{\frac{p+2}{2}}\sin\frac{p}{q}\theta.$$

(C) If p is odd, q of the form $4r+1$, and $\sin\theta$ positive,

$$C\,.\sqrt[q]{(\sin^p\theta)} = (-1)^{\frac{p-1}{2}}\sin\frac{p}{q}\theta \text{ and } S\,.\sqrt[q]{(\sin^p\theta)} = (-1)^{\frac{p-1}{2}}\cos\frac{p}{q}\theta.$$

(D) If p is odd, q of the form $4r+3$, and $\sin\theta$ negative,

$$C\,.\sqrt[q]{(-\sin\theta)^p} = (-1)^{\frac{p-1}{2}}\sin\frac{p}{q}\theta \text{ and } S\,.\sqrt[q]{(-\sin\theta)^p} = (-1)^{\frac{p-1}{2}}\cos\frac{p}{q}\theta.$$

In (A), (B), (C), (D), θ is the angle nearest to $\frac{1}{2}q\pi$.

625. In the following article, an important application is made of the Binomial Theorem.. The proof given is equivalent to establishing an equation involving the first and second differential coefficients of the function to be expanded.

626. *To expand* $\{(1+x^2)^{\frac{1}{2}}+x\}^n$ *in powers of* x, *where* $x<1$.

We may write

$$1+nx+a_2x^2+a_3x^3+\ldots+a_rx^r+\ldots=\{(1+x^2)^{\frac{1}{2}}+x\}^n \quad \ldots(I),$$

the first two terms being immediately derivable from the binomial theorem.

Now write $x+h$ for x. The left-hand becomes

$$1+n(x+h)+a_2(x+h)^2+\ldots+a_r(x+h)^r+\ldots$$

The right-hand becomes

$$\{(1+x^2+2hx+h^2)^{\frac{1}{2}}+x+h\}^n=\{(1+x^2)^{\frac{1}{2}}+\tfrac{1}{2}.2hx(1+x^2)^{-\frac{1}{2}}+x+h+\ldots\}^n$$

$$=\{(1+x^2)^{\frac{1}{2}}+x\}^n\{1+h(1+x^2)^{-\frac{1}{2}}+\ldots\}^n=\{(1+x^2)^{\frac{1}{2}}+x\}^n\{1+nh(1+x^2)^{-\frac{1}{2}}+\ldots\}.$$

Thus, equating the coefficients of h, after substituting from (I),

$$(1+x^2)^{\frac{1}{2}}(n+2a_2x+\ldots+ra_rx^{r-1}+\ldots)=n(1+nx+a_2x^2+\ldots+a_rx^r+\ldots)\ldots(II).$$

Again in (II) write $x+h$ for x. Then

$$\{(1+x^2)^{\frac{1}{2}}+hx(1+x^2)^{-\frac{1}{2}}+\ldots\}\{n+2a_2(x+h)+\ldots+ra_r(x+h)^{r-1}+\ldots\}$$
$$=n\{1+n(x+h)+a_2(x+h)^2+\ldots+a_r(x+h)^r+\ldots\}.$$

Equating the coefficients of h again, we have

$$(1+x^2)^{\frac{1}{2}}\{2a_2+\ldots+r(r-1)a_rx^{r-2}+\ldots\}+x(1+x^2)^{-\frac{1}{2}}(n+2a_2+\ldots+ra_rx^{r-1}+\ldots)$$
$$=n(n+2a_2x+\ldots+ra_rx^{r-1}+\ldots)$$
$$=n^2(1+x^2)^{-\frac{1}{2}}(1+nx+a_2x^2+\ldots+a_rx^r+\ldots) \text{ by (II).}$$

Multiply by $(1+x^2)^{\frac{1}{2}}$. Thus

$$(1+x^2)\{2a_2+\ldots+r(r-1)a_rx^{r-2}+\ldots\}+x(n+2a_2+\ldots+ra_rx^{r-1}+\ldots)$$
$$=n^2(1+nx+a_2x^2+\ldots+a_rx^r+\ldots) \quad \ldots\ldots\ldots(III)$$

In (III) equate the coefficients of x^r. Thus

$$(r+2)(r+1)a_{r+2}+r(r-1)a_r+ra_r=n^2a_r;$$

$$\therefore\ a_{r+2}=\frac{n^2-r^2}{(r+1)(r+2)}a_r.$$

Putting r in succession 0, 1, 2, 3.... we have

$$\{(1+x^2)^{\frac{1}{2}}+x\}^n = 1+nx+\frac{n^2}{1.2}x^2+\frac{n(n^2-1^2)}{1.2.3}x^3+\frac{n^2(n^2-2^2)}{1.2.3.4}x^4+\ldots + \frac{(n-r+2)(n-r+4)\ldots(n+r-2)}{\lfloor r}nx^r+\ldots.$$

Also substituting for a_2, a_3... in (II), we have

$$\{(1+x^2)^{\frac{1}{2}}+x\}^n = (1+x^2)^{\frac{1}{2}}\times\left[1+nx+\frac{n^2-1^2}{1.2}x^2+\frac{n(n^2-2^2)}{1.2.3}x^3+\ldots + \frac{(n-r+1)(n-r+3)\ldots(n+r-1)}{\lfloor r}x^r+\ldots\right].$$

627. In these two expansions, the multiplier of the alternate terms is $-\dfrac{r^2-n^2}{(r+1)(r+2)}x^2$, the limit of which is $-x^2$. Hence, if x is wholly real and lies between $+1$ and -1, or if x is wholly imaginary and if ix lies between $+1$ and -1, the series are convergent.

628. The above formulæ may be transformed in many ways.

If x is wholly real and less than 1, the series are *positive* (as the student may show): hence the real positive values of the fractional powers must be taken.

629. A. Let $(1+x^2)^{\frac{1}{2}}+x=z$; then $(1+x^2)^{\frac{1}{2}}-x=z^{-1}$,

and $2x=z-z^{-1}$.

If x lies between -1 and $+1$, z lies between $\sqrt{2}-1$ and $\sqrt{2}+1$; and we have

$$z^n = 1+\frac{n}{2}\left(z-\frac{1}{z}\right)+\frac{n^2}{2.4}\left(z-\frac{1}{z}\right)^2+\frac{n(n^2-1^2)}{2.4.6}\left(z-\frac{1}{z}\right)^3+\ldots$$

Hence any power of z may be expressed in ascending powers of $z-\dfrac{1}{z}$ in a series, which is convergent if z lies between $\sqrt{2}-1$ and $\sqrt{2}+1$.

It would thus appear that, by expanding the powers of $z-z^{-1}$ by the binomial theorem, we could expand any power of z

in positive and negative *integral* powers of z. And this is in a sense true. But the resulting series would be *divergent*, unless we took the terms in groups corresponding to the integral powers of $z - z^{-1}$.

630. B. Let $x = \cot\theta$; then $\sqrt{(1+x^2)} + x = \operatorname{cosec}\theta + \cot\theta = \cot\frac{1}{2}\theta$.

If then x lies between -1 and $+1$, we may take θ between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$, so that $\cot\frac{1}{2}\theta$ is positive. Thus

$$\cot^n\tfrac{1}{2}\theta = 1 + n\cot\theta + \frac{n^2}{1.2}\cot^2\theta + \frac{n(n^2-1^2)}{1.2.3}\cot^3\theta + \ldots.$$

631. *To expand* $\cos n\theta$ *and* $\sin n\theta$ *in powers of* $\sin\theta$, *when* n *has any real value.*

In the formulæ of Art. 626, put $x = i\sin\theta$. Then $\sqrt{(1+x^2)} + x = \cos\theta + i\sin\theta$. Hence the series represent some value of $(\cos\theta + i\sin\theta)^n$, i.e. of $\cos n\phi + i\sin n\phi$ where ϕ is some angle whose cosine and sine are equal to those of θ respectively.

If $\theta = 0$, $x = 0$; and the series become $= 1 = \cos n\theta$.

Hence, by the principle of continuity, if θ is *acute*,

$$1 - \frac{n^2}{\lfloor 2}\sin^2\theta + \frac{n^2(n^2-2^2)}{\lfloor 4}\sin^4\theta - \ldots = \cos n\theta;$$

$$n\sin\theta - \frac{n(n^2-1^2)}{\lfloor 3}\sin^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{\lfloor 5}\sin^5\theta - \ldots = \sin n\theta,$$

$$1 - \frac{n^2-1^2}{\lfloor 2}\sin^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{\lfloor 4}\sin^4\theta - \ldots = \frac{\cos n\theta}{\cos\theta},$$

$$n\sin\theta - \frac{n(n^2-2^2)}{\lfloor 3}\sin^3\theta + \frac{n(n^2-2^2)(n^2-4^2)}{\lfloor 5}\sin^5\theta - \ldots = \frac{\sin n\theta}{\cos\theta}.$$

The first and last of these series *terminate* only if n is an *even integer;* the middle two *terminate* only if n is an *odd integer*. For these special cases, the above formulæ were proved in Art. 375.

632. *To show that,* if n is positive, *the series, obtained by multiplying all the terms in the expansion of* $(1-1)^n$ *by any positive or negative* finite *quantities, is* convergent.

As soon as r becomes greater than n, the terms in the expansion of $(1-1)^n$ *after the* r^{th} have all the same sign.

If n is *positive*, the series is convergent by Art. 461, and is in fact equal to zero. Hence the sum of the terms *after the* r^{th} is finite, being numerically equal but opposite in sign to the sum of the terms *up to the* r^{th}.

Hence, if all the terms *after the* r^{th} are multiplied by any *positive finite* quantities, the resulting series will be *convergent* by Art. 453. And if any of the signs are then *changed*, the resulting series will still be *convergent* by Art. 445.

Hence, *the series obtained by multiplying all the terms in the expansion of* $(1-1)^n$, where n is positive, *by any positive or negative* finite *quantities is* convergent. Q. E. D.

633. *To expand* $(2\cos\theta)^n$ *in ratios of multiples of* θ, *where* n *has any* (*integral or fractional*) positive *value.*

The *general* value of $(2\cos\theta)^n$ is given by the expression

$$(2\cos\theta)^n(\cos 2\lambda\pi \,.\, n + i\sin 2\lambda\pi \,.\, n).$$

Again $(2\cos\theta)^n = \{\cos\theta + i\sin\theta + (\cos\theta + i\sin\theta)^{-1}\}^n$.

Expanding by the binomial theorem, the general value of the above is

$$(\cos\theta+i\sin\theta)^n + n(\cos\theta+i\sin\theta)^{n-2} + \tfrac{1}{2}n(n-1)(\cos\theta+i\sin\theta)^{n-4}+\ldots$$

where, though each of the expressions $(\cos\theta+i\sin\theta)^{n-2r}$ is *many-valued*, yet each is obtained from the preceding by dividing by the *single-valued* expression

$$(\cos\theta+i\sin\theta)^2, \text{ i.e. } \cos 2\theta + i\sin 2\theta.$$

Hence, by De Moivre's theorem, the above expansion

$$= \cos n\theta + n\cos(n-2)\theta + \tfrac{1}{2}n(n-1)\cos(n-4)\theta + \ldots$$
$$+ i\sin n\theta + in\sin(n-2)\theta + \tfrac{1}{2}in(n-1)\sin(n-4)\theta + \ldots$$

where θ must have *the same value throughout.*

Since n is *positive*, this series is *convergent* by the last article.

Hence also, by making a sufficiently small change in the value of θ, we may make the change in the value of the series as small as we please.

Now suppose $\theta = 2r\pi$, where r has any specified integral value. Then the series becomes

$$(\cos 2r\pi \,.\, n + i \sin 2r\pi \,.\, n)\{1 + n + \tfrac{1}{2}n(n-1) + \ldots\}$$
$$= (\cos 2r\pi \,.\, n + i \sin 2r\pi \,.\, n) \text{ (positive value of } 2^n)$$

by the binomial theorem, n being greater than -1. (Art. 462.)

Thus, if $\cos\theta = 1$, we have

$$(\cos 2r\pi \,.\, n + i \sin 2r\pi \,.\, n)\{\text{positive value of } (2\cos\theta)^n\},$$
$$= \cos n\theta + n\cos(n-2)\theta + \tfrac{1}{2}n(n-1)\cos(n-4)\theta + \ldots$$
$$+ i\sin n\theta + in\sin(n-2)\theta + \tfrac{1}{2}in(n-1)\sin(n-4)\theta + \ldots$$

where $\theta = 2r\pi$.

If then θ changes continuously from $2r\pi$ to $(2r \pm \frac{1}{2})\pi$, $\cos\theta$ remains positive, the positive value of $(2\cos\theta)^n$ also changes continuously, and the above series changes continuously. Hence the multiplier $\cos 2r\pi \,.\, n + i \sin 2r\pi \,.\, n$ must not change discontinuously into $\cos 2r'\pi \,.\, n + i \sin 2r'\pi \,.\, n$.

Thus, if $\cos\theta$ has *any* positive value, the above equation still holds if, in the series, we take θ to be the angle nearest to $2r\pi$ whose cosine $= \cos\theta$.

Next suppose $\theta = (2r+1)\pi$, where r has any specified value. Then the series becomes

$$\{\cos(2r+1)\pi \,.\, n + i \sin(2r+1)\pi \,.\, n\} \text{ (positive value of } 2^n).$$

Thus, if $\cos\theta = -1$, we have

$$\{\cos(2r+1)\pi \,.\, n + i \sin(2r+1)\pi \,.\, n\}\{\text{positive value of } (-2\cos\theta)^n\},$$
$$= \cos n\theta + n\cos(n-2)\theta + \tfrac{1}{2}n(n-1)\cos(n-4)\theta + \ldots$$
$$+ i\sin n\theta + in\sin(n-2)\theta + \tfrac{1}{2}in(n-1)\sin(n-4)\theta + \ldots$$

where $\theta = (2r+1)\pi$.

As before then, if $\cos\theta$ has *any* negative value, this equation still holds, if, in the series, we take θ to be the angle nearest to $(2r+1)\pi$ whose cosine $= \cos\theta$.

634. In the first result of the last article suppose θ is acute. Then, $r = 0$; and, equating real and imaginary terms, we have

$$\cos n\theta + n\cos(n-2)\theta + \tfrac{1}{2}n(n-1)\cos(n-4)\theta + \ldots$$

and

$$\sin n\theta + n\sin(n-2)\theta + \tfrac{1}{2}n(n-1)\sin(n-4)\theta + \ldots$$

equal, respectively, to the real positive value of $(2\cos\theta)^n$ and to 0

A. If however θ is *not* acute, but differs from $2r\pi$ by an acute angle, the above series are equal to the real positive value of $(2\cos\theta)^n$ multiplied, respectively, by $\cos 2r\pi \,.\, n$ and $\sin 2r\pi \,.\, n$.

Thus, if $n = p/q$, where q is even, we may put $2r = q$, and the above series become equal respectively to the real value of $-(2\cos\theta)^n$ and to 0.

If further $\frac{1}{2}q$ is even, we may put $2r = \frac{1}{2}q$, and the above series become equal, respectively, to 0 and to the real value of $(-1)^{\frac{p-1}{2}}(2\cos\theta)^n$.

B. If θ differs from $(2r+1)\pi$ by an acute angle, the above series are equal to the real positive value of $(-2\cos\theta)^n$ multiplied, respectively, by $\cos(2r+1)\pi \,.\, n$ and $\sin(2r+1)\pi \,.\, n$.

Thus, if $n = p/q$, where q is odd, we may put $2r+1 = q$; and the above series become equal, respectively, to $(-1)^p(-2\cos\theta)^n$ and to 0.

If q is even and $\frac{1}{2}q$ is odd, we may put $2r+1 = \frac{1}{2}q$; and they become equal, respectively, to 0 and $(-1)^{\frac{p-1}{2}}(-2\cos\theta)^n$.

635. *To extend the above results to the case, when* n *is* negative.

Let $(n)_r$ represent the coefficient of x^r in $(1+x)^n$.

Then, as shown in Art. 464,

$$(n)_{r-1} + (n)_r = (n+1)_r.$$

This formula follows at once from the fact that $(1+x)^{n+1}$ must be obtainable from $(1+x)^n$ by multiplying by $1+x$.

Now let

$$f(n) \equiv \cos n\theta + (n)_1 \cos(n-2)\theta + \ldots + (n)_r \cos(n-2r)\theta$$
$$+ i\sin n\theta + i(n)_1 \sin(n-2)\theta + \ldots + i(n)_r \sin(n-2r)\theta$$

where the coefficients are those of $(1+x)^n$ and the series terminates at $(n)_r$. Multiply by $2\cos\theta$, and resolve each product $2\cos\theta\cos p\theta$ into the sum of $\cos(p+1)\theta$ and $\cos(p-1)\theta$, and $2\cos\theta\sin p\theta$ into $\sin(p+1)\theta + \sin(p-1)\theta$. Thus, by the above relation between the coefficients of the binomial expansions,

$$2\cos\theta \,.\, f(n) = f(n+1)$$
$$+ (n)_r \cos(n-2r-1)\theta + i(n)_r \sin(n-2r-1)\theta.$$

If then, when r is indefinitely increased, (1) $f(n+1)$ remains finite, (2) $(n)_r$ remains finite, (3) $\cos\theta$ is not zero, then $f(n)$ will be finite.

Now it was shown in Art. 519 (and also in Art. 462), that, when r is infinite, $(n)_r$ is zero or infinite according as n is greater or less than -1. If $n=-1$, $(n)_r$ becomes ± 1.

If then n lies between 0 and -1, the series $f(n)$ is convergent and $=(2\cos\theta)^n$ provided $\cos\theta$ is not zero.

If $n=-1$, the series $f(n)$ is finite, but indeterminate.

If $n<-1$, the series $f(n)$ is infinite for all but certain special forms of r, when r is infinite.

636. In the expansions of the ratios of $n\theta$ in terms of the ratios of θ, we may, by putting $n=\infty$, obtain expansions of the ratios of θ in terms of θ. This has been done in Art. 501.

637. But, by putting $n=0$, we obtain expansions of θ in terms of the ratios of θ.

The results are true, in each case, only if θ represents the circular measure of an *acute* angle.

Thus, from Art. 619, we have, if $\tan\theta<1$,

$$\tan\theta-\frac{(n-1)(n-2)}{1.2.3}\tan^3\theta+\frac{(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}\tan^5\theta-\ldots$$
$$=\frac{\sin n\theta}{n\cos^n\theta},$$

$$-\frac{n-1}{2}\tan^2\theta+\frac{(n-1)(n-2)(n-3)}{1.2.3.4}\tan^4\theta-\ldots=\frac{\cos n\theta-\cos^n\theta}{n\cos^n\theta}.$$

Putting $n=0$, and observing that the expansion of $\cos^n\theta$ in powers of n gives $1+n\log\cos\theta+\frac{1}{2}(n\log\cos\theta)^2+\ldots$, we have

$$\tan\theta-\tfrac{1}{3}\tan^3\theta+\tfrac{1}{5}\tan^5\theta-\ldots=\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\tfrac{1}{2}\tan^2\theta-\tfrac{1}{4}\tan^4\theta+\tfrac{1}{6}\tan^6\theta-\ldots=\log\sec\theta \quad \ldots\ldots(2).$$

[The latter result follows at once from the equation

$$\sec\theta=(1+\tan^2\theta)^{\frac{1}{2}}.]$$

Again, from Art. 631, we have

$$\sin\theta - \frac{n^2-1^2}{\underline{|3}}\sin^3\theta + \frac{(n^2-1^2)(n^2-3^2)}{\underline{|5}}\sin^5\theta - \ldots = \frac{\sin n\theta}{n},$$

$$\frac{1}{\underline{|2}}\sin^2\theta - \frac{n^2-2^2}{\underline{|4}}\sin^4\theta + \frac{(n^2-2^2)(n^2-4^2)}{\underline{|6}}\sin^6\theta - \ldots = \frac{1-\cos n\theta}{n^2}.$$

Putting $n=0$, we have

$$\sin\theta + \frac{1}{2}\frac{\sin^3\theta}{3} + \frac{1.3}{2.4}\frac{\sin^5\theta}{5} + \ldots = \theta \quad \ldots\ldots\ldots\ldots(3),$$

$$\frac{\sin^2\theta}{2} + \frac{2}{3}\frac{\sin^4\theta}{4} + \frac{2.4}{3.5}\frac{\sin^6\theta}{6} + \ldots = \frac{\theta^2}{2} \quad \ldots\ldots\ldots\ldots(4).$$

Again, from Art. 634, we have

$$\sin(2-n)\theta + \frac{n-1}{1.2}\sin(4-n)\theta + \frac{(n-1)(n-2)}{1.2.3}\sin(6-n)\theta + \ldots = \frac{\sin n\theta}{n},$$

$$\cos(2-n)\theta + \frac{n-1}{1.2}\cos(4-n)\theta + \frac{(n-1)(n-2)}{1.2.3}\cos(6-n)\theta + \ldots = \frac{(2\cos\theta)^n - \cos n\theta}{n}.$$

Putting $n=0$, we have

$$\sin 2\theta - \tfrac{1}{2}\sin 4\theta + \tfrac{1}{3}\sin 6\theta - \ldots = \theta \quad \ldots\ldots\ldots\ldots(5),$$

$$\cos 2\theta - \tfrac{1}{2}\cos 4\theta + \tfrac{1}{3}\cos 6\theta - \ldots = \log 2\cos\theta \quad \ldots\ldots(6).$$

Other results may be obtained on the same principle.

§ 3. Complex or Imaginary Indices.

638. The use of complex expressions of the form $a+bi$, where a and b are real, is explained for symbolical purposes by the understanding that the equation $a+bi=c+di$ shall be interpreted to mean that $a=c$ and $b=d$.

It remains then to examine what use can be made of imaginary or complex *indices*.

639. DEF. Let $f(x)$ be some single-valued algebraical function of x, not containing x as an index, such that for any complex values of x and y,

$$f(x) \times f(y) = f(x+y),$$

then, if $$f(x) \equiv X \text{ and } f(1) \equiv a,$$

X is said to be a value of the x^{th} power of a; and x is said to be a logarithm of X to base a.

Thus the several *forms* of $f(x)$ which will satisfy the conditions that $f(1) = a$, and $f(x) \times f(y) = f(x+y)$, will be the several values of a^x.

And the several *values* of x, which will satisfy the equation $f(x) = X$ for any of the above forms of f, will be the values of $\log_a X$ for that particular form of f.

640. *To find the forms of* f(x).

Suppose that

$$f(x) \equiv c_0 + cx + c_2x^2 + c_3x^3 + \ldots$$

Since the relation $f(x) \times f(y) = f(x+y)$ requires that $f(0) = 1$, we see that $c_0 = 1$. The same relation gives

$$1 + c(x+y) + c_2(x+y)^2 + \ldots + c_r(x+y)^r + \ldots$$
$$= (1 + cx + c_2x^2 + \ldots + c_{r-1}x^{r-1} + \ldots)(1 + cy + \ldots).$$

By multiplying out the term $c_r(x+y)^r$, the coefficient of y is clearly rc_rx^{r-1}. Hence equating the coefficients of yx^{r-1},

$$rc_r = c \,.\, c_{r-1}, \quad \therefore\ c_r = \frac{c \,.\, c_{r-1}}{r}.$$

Putting r successively equal to 1, 2, 3,... we have

$$c_r = \frac{c^r}{\lfloor r}.$$

Hence

$$f(x) \equiv 1 + cx + \frac{c^2x^2}{\lfloor 2} + \frac{c^3x^3}{\lfloor 3} + \ldots + \frac{c^rx^r}{\lfloor r} + \ldots$$

The coefficient c is obtained from the condition that $f(1)$

shall have any given value a; i.e. c is any solution of the equation

$$1 + c + \frac{c^2}{\lfloor 2} + \frac{c^3}{\lfloor 3} + \ldots + \frac{c^r}{\lfloor r} + \ldots = a.$$

If c is any root of this equation, the several values of a^x are the several corresponding values of

$$1 + cx + \frac{c^2x^2}{\lfloor 2} + \frac{c^3x^3}{\lfloor 3} + \ldots$$

And the several values of $\log_a X$ for each value of c, are the several values of x found from the equation

$$1 + cx + \frac{c^2x^2}{\lfloor 2} + \frac{c^3x^3}{\lfloor 3} + \ldots = X.$$

641. *To find the nature of the roots of the equation* $f(x) = 1$, where

$$f(x) \equiv 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \ldots + \frac{x^r}{\lfloor r} + \ldots$$

I. $f(x) = 1$ has no *real* root, except $x = 0$. For, if x is real, $f(x)$ is the real positive value of $\{f(1)\}^x$. But $f(1) > 1$. Hence all real *positive* powers of $f(1)$ are > 1; and all real *negative* powers of $f(1)$ are < 1. Hence $f(x)$ cannot $= 1$, for any real value of x, other than zero.

II. $f(x) = 1$ has no *complex* root.

For, since the coefficients in $f(x)$ are real, if $a + bi$ were a root of $f(x) = 1$, $a - bi$ would be a root.

But $f(a + bi) \times f(a - bi) = f(2a)$.

If then both $f(a + bi)$ and $f(a - bi)$ were equal to 1, $f(2a)$ would be equal to 1; $\therefore$ by I., a must be zero.

Hence all the roots of $f(x) = 1$ are *wholly imaginary*.

III. Any integral multiple of the common measure of any two roots of $f(x) = 1$ must also be a root.

Let $r\mu i$ and $r'\mu i$ be two roots having the common measure μi, so that r and r' are integers.

Then, since $f(x) \times f(y) = f(x+y)$, and since $f(r\mu i) = 1$ and $f(r'\mu i) = 1$, therefore

$$f(pr \sim p'r')\,\mu i = 1, \text{ where } p \text{ and } p' \text{ are any integers.}$$

But, by the theory of numbers, we may make $pr \sim p'r' = 1$.

Hence μi is a root of $f(x) = 1$; and $\therefore$ any multiple of μi is a root.

IV. All the roots of $f(x) = 1$ must be integral multiples of some common root.

For, if μi and any root *outside* the series $r\mu i$ have any common measure, all the multiples of this common measure will be roots. And these will include the series $r\mu i$.

Hence, as long as we take two *commensurable* roots, all the roots will be included in the multiples of their common measure.

But, if two roots λi and $\lambda' i$ were *incommensurable*, we could find a fraction $\frac{p}{q}$ as near as we please to $\frac{\lambda}{\lambda'}$; so that $p\lambda' i - q\lambda i$, which would also be a root, could be made as small as we please. And since any multiple of this root would be a root, it would follow that any wholly imaginary value whatever would be a root. That is, writing $x = iy$, the equations

$$1 - \frac{y^2}{\lfloor 2} + \frac{y^4}{\lfloor 4} - \frac{y^6}{\lfloor 6} + \ldots = 1$$

and

$$y - \frac{y^3}{\lfloor 3} + \frac{y^5}{\lfloor 5} - \frac{y^7}{\lfloor 7} + \ldots = 0$$

would be true for *all real* values whatever. But this is impossible since the coefficients of these equations are not zero.

Hence

The roots of $f(x) = 1$ *are all the* positive and negative integral multiples *of some wholly imaginary root which we may call* $2\pi i$.*

* The introduction of the symbol π will be justified later by showing that it has here its usual value.

642. *To find the general form of the values of* $\log_a b$.

The general values of $\log_a b$ are the roots of

$$1 + cx + \frac{c^2x^2}{\lfloor 2} + \frac{c^3x^3}{\lfloor 3} + \ldots = b \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where c is any root of

$$1 + c + \frac{c^2}{\lfloor 2} + \frac{c^3}{\lfloor 3} + \ldots = a.$$

Let $\lambda_e a$ be any one of the roots of this equation: and $\lambda_e a + k$ any other root. Then, the left hand being $f(c)$, we have

$$f(\lambda_e a) \times f(k) = f(\lambda_e a + k);$$

$$\therefore \quad a \times f(k) = a, \qquad \therefore \quad f(k) = 1.$$

$\therefore$ k is any solution of the equation $f(x) = 1$; i.e. $k = 2n\pi i$.

$\therefore$ the roots of $f(x) = a$ are $\lambda_e a + 2n\pi i$,

and the roots of $f(x) = b$ are $\lambda_e b + 2m\pi i$.

Hence the roots of (1) are $\dfrac{\lambda_e b + 2m\pi i}{c}$, where $c = \lambda_e a + 2n\pi i$.

That is,

the general values of $\log_a b$ **are** $\dfrac{\lambda_e b + 2m\pi i}{\lambda_e a + 2n\pi i}$.

This means that there is some function of x, viz.:

$$\phi(x) \equiv 1 + (\lambda_e a + 2n\pi i)\, x + \frac{(\lambda_e a + 2n\pi i)^2}{\lfloor 2} x^2 + \ldots$$

such that

(1) $\phi(x) \times \phi(y) = \phi(x + y)$.

(2) $\phi(1) = a$.

(3) $\phi(x) = b$, if $x = \dfrac{\lambda_e b + 2m\pi i}{\lambda_e a + 2n\pi i}$,

where $\lambda_e(t)$ is any solution of $1 + x + \dfrac{x^2}{\lfloor 2} + \dfrac{x^3}{\lfloor 3} + \ldots = t$.

Particular cases of this result should be carefully noted. Thus we will put

$$1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{4}{\lfloor 4} + \ldots = e.$$

Then for a write e. Then one value of $\lambda_e e$ is 1. Thus

$$\text{the general values of } \log_e b \text{ are } \frac{\lambda_e b + 2m\pi i}{1 + 2n\pi i}.$$

Or again, put $b = 1$. Then one value of $\lambda_e b$ is 0. Thus

$$\text{the general values of } \log_a 1 \text{ are } \frac{2m\pi i}{\lambda_e a + 2n\pi i}.$$

Here put $a = e$. Thus

$$\text{the general values of } \log_e 1 \text{ are } \frac{2m\pi i}{1 + 2n\pi i}.$$

Putting m and n equal to 0, we see that $\lambda_e b$ is a special value of $\log_e b$. Putting $n = 0$, we see that $2m\pi i$ is a value of $\log_e 1$.

And in general we find:

Each logarithm, such as $\log_a b$, *has a double infinity of values, which is obtained by giving a single infinity of values to* $\log_e b$ *and to* $\log_e a$.

643. *To find the general values of* a^x.

The expansion $$1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \dots$$

may be called the **expansional value** of e^x.

Thus, since the general value of a^x is

$$1 + cx + \frac{c^2x^2}{\lfloor 2} + \frac{c^3x^3}{\lfloor 3} + \dots$$

where $c = \lambda_e a + 2n\pi i$; we have

The **general values** of $\mathbf{a^x}$ are the **expansional values** of $\mathbf{e^{(\lambda_e a + 2n\pi i)x}}$, for all **integral** values of n.

Assuming then that a may be complex, we will apply this to the three cases according as x is wholly real, wholly imaginary, or complex.

(1) Let x be wholly real $= p/q$ say.

Then the expansional values of $e^{(\lambda_e a + 2n\pi i)p/q}$ are those of

$$e^{\frac{p}{q}\lambda_e a} \times e^{\frac{p}{q}.2n\pi i}.$$

Here putting $n = 0, 1, 2 \ldots q-1$, we get q different values, since $2\pi i$ is the smallest value of x for which the expansional value of e^x is 1. After this the values recur. Hence

If $x = \frac{p}{q}$; p and q being real integers prime to one another, a^x has q different values; which form a geometrical progression, whose common ratio is a q^{th} root of unity, viz. $e^{\frac{2p\pi i}{q}}$.

(2) Let x be wholly imaginary $= yi$.

Then the expansional values of $e^{(\lambda_a a + 2n\pi i)yi}$ are those of $e^{iy\lambda_a a} \cdot e^{-2ny\pi}$. This gives a different value for every value of n.

Hence, if x is wholly imaginary, a^x has an infinite number of values, which form a geometrical progression whose common ratio is the expansional value of $e^{2x\pi i}$.

(3) Let x be complex $= y + zi$.

Then the expansional values of $e^{(\lambda_a a + 2n\pi i)(y+zi)}$ are those of $e^{y\lambda_a a} \cdot e^{iz\lambda_a a} \cdot e^{-2nz\pi} \cdot e^{2ny\pi i}$. The factor $e^{-2nz\pi}$ has an infinite number of values in geometric progression: and the number of values of $e^{2ny\pi i}$ is equal to the denominator of y.

644. *To trace the changes in the* expansional value of e^{xi}, *where* x *is real.*

Writing $f(xi) \equiv$ the expansional value of e^{xi},
we have

$$f(xi) \times f(yi) = f\{(x+y)\,i\}.$$

Let $\chi(x)$ and $i\psi(x)$ be the real and imaginary parts of $f(xi)$.

Then

$$\{\chi(x) + i\psi(x)\}\{\chi(y) + i\psi(y)\} = \chi(x+y) + i\psi(x+y).$$

Hence equating the real and imaginary terms

$$\chi(x+y) = \chi(x) \cdot \chi(y) - \psi(x) \cdot \psi(y) \quad \ldots\ldots\ldots\ldots(1),$$
$$\psi(x+y) = \psi(x) \cdot \chi(y) + \chi(x) \cdot \psi(y) \quad \ldots\ldots\ldots\ldots(2).$$

Now since $\chi(x)$ contains only *even* and $\psi(x)$ only *odd* powers of x,

$$\therefore \quad \chi(-x) = \chi(x) \text{ and } \psi(-x) = -\psi(x) \quad \ldots\ldots(3),$$

also $$\chi(0) = 1 \quad \text{and} \quad \psi(0) = 0 \quad \ldots\ldots(4),$$

and $$\chi(x-y) = \chi(x) \cdot \chi(y) + \psi(x) \cdot \psi(y) \quad \ldots\ldots(5),$$

$$\psi(x-y) = \psi(x) \cdot \chi(y) - \chi(x) \cdot \psi(y) \quad \ldots\ldots(6).$$

From (1), (2), (5), and (6) we have

$$\chi(y) - \chi(x) = 2\psi\{\tfrac{1}{2}(x+y)\} \cdot \psi\{\tfrac{1}{2}(x-y)\} \quad \ldots\ldots(7),$$

$$\psi(x) - \psi(y) = 2\chi\{\tfrac{1}{2}(x+y)\} \cdot \psi\{\tfrac{1}{2}(x-y)\} \quad \ldots\ldots(8).$$

Putting in (5) $y = x$, we have

$$1 = \{\chi(x)\}^2 + \{\psi(x)\}^2 \quad \ldots\ldots(9).$$

Hence $\chi(x)$ and $\psi(x)$ are arithmetically not greater than 1.

Now $f(2\pi i) = 1$, $\therefore f(xi + 2\pi i) = f(xi)$; i.e. the values of $f(xi)$ recur whenever x is increased by 2π.

Again, since

$$f(2\pi i) = 1, \ \therefore f(\pi i) = \sqrt{1} = +1 \text{ or } -1.$$

But $f(\pi i)$ cannot be $+1$, because, by definition, 2π is the smallest value of x for which $f(xi) = 1$.

$$\therefore \quad f(\pi i) = -1; \qquad \text{i.e. } \chi(\pi) = -1 \text{ and } \psi(\pi) = 0,$$

$$\therefore \chi(-\pi) = -1 \text{ and } \psi(-\pi) = 0.$$

$$\therefore \quad f(xi + \pi i) = -f(xi);$$

$$\text{i.e.} \quad \chi(x+\pi) = -\chi(x) \text{ and } \psi(x+\pi) = -\psi(x).$$

$$\therefore \quad \chi(\pi - x) = -\chi(x) \text{ and } \psi(\pi - x) = \psi(x).$$

Again, since

$$f(\pi i) = -1, \qquad \therefore f(\tfrac{1}{2}\pi i) = \sqrt{(-1)} = +i \text{ or } -i;$$

so that $$\chi(\tfrac{1}{2}\pi) = 0 \text{ and } \psi(\tfrac{1}{2}\pi) = +i \text{ or } -i.$$

We must show that, from $x = 0$ to $x = \frac{1}{2}\pi$, $\chi(x)$ and $\psi(x)$ are both *positive*. If x is a small positive quantity, $\chi(x)$ and $\psi(x)$ are clearly positive. Hence, being continuous, they cannot become negative without passing through the value zero.

If possible, let $\chi(y)$ or $\psi(y) = 0$ for some value of x between 0 and $\frac{1}{2}\pi$. Then, since

$$\{\chi(y)\}^2 + \{\psi(y)\}^2 = 1 \text{ and } f(yi) = \chi(y) + i\psi(y),$$

$f(yi)$ would equal ± 1 or $\pm i$. Hence $f(2yi)$ would be ± 1 and $f(4yi)$ would be 1. But $f(xi)$ is not 1, until $x = 2\pi$, $\therefore$ neither $\psi(y)$ nor $\chi(y)$ can vanish if y is between 0 and $\frac{1}{2}\pi$.

$\therefore$ $\psi(x)$ and $\chi(x)$ are always positive from $x = 0$ to $x = \frac{1}{2}\pi$.

$\therefore$ $f(\frac{1}{2}\pi i) = +i$; i.e. $\chi(\frac{1}{2}\pi) = 0$ and $\psi(\frac{1}{2}\pi) = 1$.

$\therefore$ $\chi(-\frac{1}{2}\pi) = 0$ and $\psi(-\frac{1}{2}\pi) = -1$.

$\therefore$ $f(xi + \frac{1}{2}\pi i) = if(xi)$;

i.e. $\chi(x + \frac{1}{2}\pi) = -\psi(x)$ and $\psi(x + \frac{1}{2}\pi) = \chi(x)$,

$\therefore$ $\chi(\frac{1}{2}\pi - x) = \psi(x)$ and $\psi(\frac{1}{2}\pi - x) = \chi(x)$.

$\therefore$ $\chi(\frac{1}{4}\pi) = \psi(\frac{1}{4}\pi) = \dfrac{1}{\sqrt{2}}$ since $\{\chi(x)\}^2 + \{\psi(x)\}^2 = 1$: and so on.

Finally, by (7) and (8), as long as $\frac{1}{2}(x+y)$ and $\frac{1}{2}(x-y)$ lie between 0 and $\frac{1}{2}\pi$, $\psi(x)$ must *increase* and $\chi(x)$ must *decrease* as x increases.

Thus, between 0 and $\frac{1}{2}\pi$, $\psi(x)$ continually increases from 0 to 1; and $\chi(x)$ continually decreases from 1 to 0.

Hence the values of $\psi(x)$ and $\chi(x)$ cannot be repeated as x passes from 0 to $\frac{1}{2}\pi$.

645. *To find a value for* $\lambda_e(1+y)$, *i.e. a root of the equation*

$$1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots = 1 + y.$$

The value of $\lambda_e(1+y)$ has to satisfy the condition

$$\lambda_e(1+y) + \frac{\{\lambda_e(1+y)\}^2}{\lfloor 2} + \frac{\{\lambda_e(1+y)\}^3}{\lfloor 3} + \ldots = y;$$

$\therefore$ when $\lambda_e(1+y) = 0$, $y = 0$ and $\dfrac{y}{\lambda_e(1+y)} = 1$.

If then $\lambda_e(1+y) \equiv c_0 + c_1 y + c_2 y^2 + c_3 y^3 + \ldots$,

we must have $c_0 = 0$, and $c_1 = 1$; so that

$$\lambda_e(1+y) = y + c_2 y^2 + c_3 y^3 + \ldots + c_r y^r + \ldots$$

Writing $f(x)$ for the series $1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$, we have

$$f(x) \times f(x') = f(x + x');$$

$\therefore$ if a is a root of $f(x) = 1 + y$; and a' of $f(x) = 1 + z$; then $a + a'$ is a root of $f(x) = (1 + y)(1 + z) = 1 + y + (1 + y)z$.

That is, we may write

$$\lambda_e(1 + y) + \lambda_e(1 + z) = \lambda_e\{1 + y + (1 + y)z\};$$

$$\therefore \lambda_e(1 + y) + z + c_2z^2 + \ldots = \{y + (1 + y)z\} + \ldots + c_r\{y + (1 + y)z\}^r + \ldots$$

Equating the coefficients of z, we have

$$1 = 1 + y + \ldots + rc_r y^{r-1}(1 + y) + (r + 1)c_{r+1}y^r(1 + y) + \ldots$$

Equating to zero the coefficient of y^r, we have

$$rc_r + (r + 1)c_{r+1} = 0; \quad \therefore \quad c_{r+1} = \frac{-r}{r+1}c_r.$$

Putting r successively equal to 1, 2, 3... we have

$$c_{r+1} = \frac{(-1)^r}{r+1};$$

$$\therefore \quad \lambda_e(1 + y) = y - \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4 + \ldots$$

which is a convergent series if mod. y is not greater than 1. [For, by Art. 596, mod. y^n = (mod. y)^n. See also Arts. 616, 453, 445.]

We must here distinguish the cases according as y is wholly real, wholly imaginary, or complex.

(1) Let y be wholly real.

Then the series for $\lambda_e(1 + y)$ is real;

$\therefore$ $y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \ldots$ = the one real value of $\log_e(1 + y)$.

(2) Let y be wholly imaginary $= zi$ say.

Then $\lambda_e(1 + zi) = i(z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \ldots) + \frac{1}{2}(z^2 - \frac{1}{2}z^4 + \frac{1}{3}z^6 - \ldots)$

$$= i(z - \tfrac{1}{3}z^3 + \tfrac{1}{5}z^5 - \ldots) + \tfrac{1}{2}\lambda_e(1 + z^2).$$

$$\therefore 2i(z - \tfrac{1}{3}z^3 + \tfrac{1}{5}z^5 - \ldots) = 2\lambda_e(1 + zi) - \lambda_e(1 + z^2) = \lambda_e\frac{1 + zi}{1 - zi}.$$

Thus $\lambda_e\dfrac{1 + zi}{1 - zi}$ is wholly imaginary. Now when z equals 0, the series $= 0$; hence as z varies from 0 to 1 or from 0 to -1, the series changes continuously.

Now between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$ there is only *one* value of x for which the expansion of e^{ix} has any given value.

Hence the series $2i(z-\frac{1}{3}z^3+\frac{1}{5}z^5-\ldots)$ = the value of $\lambda_e\frac{1+zi}{1-zi}$ which lies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

$\therefore \lambda_e(1+zi)=\frac{1}{2}$ the real value of $\log_e(1+z^2)$

+ the value of $\frac{1}{2}\lambda_e\frac{1+zi}{1-zi}$ which lies between $-\frac{1}{4}\pi$ and $+\frac{1}{4}\pi$.

(3) Let y be complex $=y+zi$ say. Then

$$\lambda_e(1+y+zi)=\tfrac{1}{2}\lambda_e(1+2y+y^2+z^2)+\tfrac{1}{2}\lambda_e\frac{1+zi/(1+y)}{1-zi/(1+y)},$$

which reduces to cases (2) and (3).

646. Hence we can find an expansion for π.

Thus $f(\frac{1}{2}\pi i)=i=\dfrac{1+i}{1-i}$. $\quad\therefore \lambda_e\dfrac{1+i}{1-i}=\frac{1}{2}\pi i$.

Hence putting $z=1$ in (2) of last article,

$$\tfrac{1}{4}\pi=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots$$

Other expansions for $\frac{1}{4}\pi$ may be found by observing that

$$\lambda_e\frac{1+x}{1-x}+\lambda_e\frac{1+y}{1-y}=\lambda_e\frac{1+x+y+xy}{1-x-y+xy}.$$

Thus putting $x=\frac{1}{2}i$ and $y=\frac{1}{3}i$, we have

$$\lambda_e\frac{1+\frac{1}{2}i}{1-\frac{1}{2}i}+\lambda_e\frac{1+\frac{1}{3}i}{1-\frac{1}{3}i}=\lambda_e\frac{1+i}{1-i}=\tfrac{1}{2}\pi i.$$

647. The indeterminateness of the powers and logarithms of quantities, which arise from our introducing imaginary indices, gives rise to some difficulties in the management of equations.

Thus given that $a=b$, we have

The general values of a^x are the expansional values of $e^{(\lambda_e b+2n\pi i)x}$, for the several integral values of n.

And the general values of $\log_e a$ are $\dfrac{\lambda_e b+2n\pi i}{1+2m\pi i}$.

In these expressions $\lambda_e b$ means any solution of the equation obtained by equating the expansional value of e^x to b: i.e. $\lambda_e b$ represents *some* logarithm of b to the base e.

648. The indeterminateness of these operations may be diminished however by the following conventions:—

(1) The written symbol of the form e^x shall mean *the expansional value of* e^x.

(2) The written symbol of the form $\log_e a$ shall mean any value of x for which *the expansional value of* $e^x = a$.

Then from the equation $a = b$, we should deduce

$$a^x = e^{(\log_e b + 2n\pi i)x}$$

and $$\log_e a = \log_e b + 2n\pi i.$$

In this way the logarithm to the base e would be reduced to a single indeterminateness. But those to any other base would have to remain doubly indeterminate.

649. Illustrations may be given of the confusions arising from neglecting the nature of the indeterminateness of powers and logarithms.

(1) We have $1^i = e^{i \log_e 1} = e^0 = 1$.

But $e^{2\pi i} = 1$, $\therefore (e^{2\pi i})^i = 1^i$; i.e. $e^{-2\pi} = 1$.

This result is apparently absurd.

It must be amended as follows:

$1^i = e^{i \log 1}$ = the expansional value of $e^{i(0+2n\pi i)}$

= the expansional value of $e^{-2n\pi}$.

This is an indeterminate result one of whose values is 1, viz. when $n = 0$.

Next we have a value of $e^{2\pi i}$ is 1; $\therefore$ a value of $e^{-2\pi}$ is $e^{-2n\pi}$ for some value of n; i.e. the expansional value of $e^{-2\pi}$ is the expansional value of $e^{-2n\pi}$ if $n = 1$.

(2) We have $e^{2m\pi i} = 1$; $\therefore e^{1+2m\pi i} = e$.

Raise both sides to the power $1 + 2n\pi i$, then

$$e^{1-4mn\pi^2+2(m+n)\pi i} = e^{1+2n\pi i} = e,$$

for any integral values of m and n.

This process might be repeated *ad infinitum.*

It must be amended as follows:

For simplicity we may understand that e^x means the expansional value of e^x. Then we may write $e^{1+2m\pi i} = e$, for any *assigned* value of m.

Raising both sides of this equation to the power $1 + 2n\pi i$ and still understanding expansional values, the left hand becomes $e^{(1+2m\pi i+2m'\pi i)(1+2n\pi i)}$ where m' is *unassigned.*

Assigning to m' its several values we obtain the several values required. Amongst these, viz. when $m' = -m$, we have the value e.

Applications to Trigonometrical Formulae.

650. Having worked out a purely algebraical theory of imaginary indices or logarithms we may, by means of De Moivre's Theorem, apply the results to Trigonometry.

651. *To find a series for* $\cos A + i \sin A$.

By De Moivre's Theorem, if A is a real angle and x a real quantity, the general values of $(\cos A + i \sin A)^x$ are given by the expression $\cos x(A + n \,.\, 360°) + i \sin x\,(A + n \,.\, 360°)$, where n is any integer.

But, by the theorems above discussed, the general values of $(\cos A + i \sin A)^x$ are the values of

$$1 + cx + \frac{c^2x^2}{\lfloor 2} + \frac{c^3x^3}{\lfloor 3} + \ldots$$

where c is any root of the equation

$$1 + c + \frac{c^2}{\lfloor 2} + \frac{c^3}{\lfloor 3} + \ldots = \cos A + i \sin A \ldots\ldots\ldots\ldots (1).$$

Hence, if c satisfies (1),

$$\cos x\,(A + n\,.\,360°) + i \sin x\,(A + n\,.\,360°) = 1 + cx + \frac{c^2x^2}{\underline{|2}} + \ldots \quad (2).$$

Now x, being fractional, may be indefinitely diminished. In the limit

$$\frac{\cos x\,(A + n\,.\,360°) - 1}{x} = -\frac{2\sin^2 \tfrac{1}{2}x\,(A + n\,.\,360°)}{x} = 0,$$

and $$\frac{\sin x\,(A + n\,.\,360°)}{x} = \text{circular measure of } A + n\,.\,360°;$$

$\therefore$ from (2), $i \times$ (circular measure of $A + n\,.\,360°$) $= c$.

Let θ and 2π be the circular measure of A and 360° respectively. Then by (1),

The expansional value of

$$e^{i(\theta + 2n\pi)} = \cos\theta + i \sin\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

This result justifies the introduction of the symbol π in Art. 641.

652. If we understand by e^x its expansional value, we have

$$\cos\theta + i \sin\theta = e^{i\theta} \text{ and } \therefore \cos\theta - i \sin\theta = e^{-i\theta}.$$

$$\therefore \cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) \text{ and } i \sin\theta = \tfrac{1}{2}(e^{i\theta} - e^{-i\theta}).$$

These are usually called the **Exponential values** of the sine and cosine.

It would be more correct to call $\cos\theta$ and $i \sin\theta$ the **Expansional values** of $\frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\frac{1}{2}(e^{i\theta} - e^{-i\theta})$ respectively. For these latter are at first sight *uninterpretable*, and are in any case *indeterminate;* while $\cos\theta$ and $\sin\theta$ are both interpretable and determinate.

653. We have already (Art. 508) given the following definitions:

$\cosh\theta =$ the expansional value of $\frac{1}{2}(e^{\theta} + e^{-\theta})$, and

$\sinh\theta =$ the expansional value of $\frac{1}{2}(e^{\theta} - e^{-\theta})$.

Hence we may introduce *imaginary angles*, by the definitions:—

$$\cos i\theta = \cosh\theta \text{ and } \sin i\theta = i\sinh\theta;$$

which will result from our writing $i\theta$ for θ in the values of $\cos\theta$ and $i\sin\theta$.

Thus a *wholly imaginary* angle has a *wholly real cosine*, but a *wholly imaginary sine*.

654. *To find the expansion for* $\tan^{-1}x$.

We have

$$\cos\theta + i\sin\theta = e^{i\theta} \text{ and } \cos\theta - i\sin\theta = e^{-i\theta}.$$

$\therefore$ by division,
$$\frac{1 + i\tan\theta}{1 - i\tan\theta} = e^{2i\theta}.$$

But since here e^x means the expansional value of e^x, we may use $\log_e a$ to mean a value of x for which the expansional value of e^x is a. Hence, if $\tan\theta < 1$,

$$2i\theta = \log_e(1 + i\tan\theta) - \log_e(1 - i\tan\theta),$$
$$= 2n\pi i + 2i(\tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta \ldots);$$
$$\therefore\ \theta = n\pi + \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \ldots$$

But the series after $n\pi$ is equal to the value of θ between $-\frac{1}{4}\pi$ and $+\frac{1}{4}\pi$ by Art. 645. Or, immediately by the principle of continuity, since $\tan 0 = 0$; we have

$$x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \ldots$$

= the circular measure of the acute value of $\tan^{-1}x$.

655. The exponential expressions for the sine and cosine enable us to use very compendious proofs of propositions (which, of course, are otherwise provable).

They have two important uses:

(1) In the *interpretation* of functions involving complex expressions: i.e. the reduction of these functions to the form of a complex expression.

(2) In the *summation of series* and conversely the development of functions into series.

656. Example of the interpretation of a complex function.

Reduce $(a+bi)^{p+qi}$ to the form $A+Bi$.

Let

$a+bi = r(\cos\theta + i\sin\theta)$, so that $r^2 = a^2+b^2$ and $\tan\theta = b/a$.

Then

$$(a+bi)^{p+qi} = r^{p+qi}\,.\,e^{(p+qi)\theta i} = r^p\,.\,e^{-q\theta}\,.\,r^{qi}\,e^{p\theta i}$$

$$= r^p\,.\,e^{-q\theta}\,.\,e^{(q\log r + p\theta)i}$$

$$= r^p\,.\,e^{-q\theta}\{\cos(q\log r + p\theta) + i\sin(q\log r + p\theta)\}.$$

$\therefore\ A = r^p\,.\,e^{-q\theta}\cos(q\log r + p\theta)$ and $B = r^p\,.\,e^{-q\theta}\sin(q\log r + p\theta)$.

The indeterminateness of this result should be examined.

We take r to be the *positive* value of $\sqrt{(a^2+b^2)}$: so that to $\log r$ we may give its real value. Then θ is the angle whose cosine is b/r and whose sine is a/r. Hence θ is indeterminate—its values differing by 2π. Hence the magnitude of the coefficient $e^{-q\theta}$ has an infinite number of values; and the expression $\cos(q\log r+p\theta)+i\sin(q\log r+p\theta)$ has values equal in number to the denominator of p.

657. Example of the *Summation of Series*.

To find the two sums:—

$$C \equiv 1 + x\cos\theta + x^2\cos 2\theta + x^3\cos 3\theta + \ldots$$

$$S \equiv 0 + x\sin\theta + x^2\sin 2\theta + x^3\sin 3\theta + \ldots$$

We have

$$C + Si \equiv 1 + x(\cos\theta + i\sin\theta) + x^2(\cos 2\theta + i\sin 2\theta) + \ldots$$

$$\equiv 1 + xe^{\theta i} + x^2e^{2\theta i} + x^3e^{3\theta i} + \ldots$$

$$\equiv \frac{1}{1-xe^{\theta i}} \equiv \frac{1}{1-x(\cos\theta+i\sin\theta)} \equiv \frac{1-x\cos\theta+ix\sin\theta}{(1-x\cos\theta)^2 - i^2x^2\sin^2\theta}.$$

$$\therefore\ C \equiv \frac{1-x\cos\theta}{1-2x\cos\theta+x^2} \text{ and } S = \frac{x\sin\theta}{1-2x\cos\theta+x^2}.$$

658. Example of *Development in a Series*.

To expand $\log_e(1-2x\cos\theta+x^2)$ in powers of x.

$$\mathrm{Log}_e(1-2x\cos\theta+x^2) = \log_e(1-xe^{\theta i}-xe^{-\theta i}+x^2) = \log_e(1-xe^{\theta i})(1-xe^{-\theta i})$$

$$= -x(e^{\theta i}+e^{-\theta i}) - \tfrac{1}{2}x^2(e^{2\theta i}+e^{-2\theta i}) - \tfrac{1}{3}x^3(e^{3\theta i}+e^{-3\theta i}) - \ldots$$

$$\therefore\ -\tfrac{1}{2}\log_e(1-2x\cos\theta+x^2) = x\cos\theta + \tfrac{1}{2}x^2\cos 2\theta + \tfrac{1}{3}x^3\cos 3\theta + \ldots$$

659. To expand θ in powers of x, when $\sin\theta = x\sin(\theta+\alpha)$.

Here $e^{\theta i} - e^{-\theta i} = xe^{(\theta+\alpha)i} - xe^{-(\theta+\alpha)i}$,

$\therefore\ e^{2\theta i} = \dfrac{1-xe^{-\alpha i}}{1-xe^{\alpha i}}$,

$\therefore$ (the acute value of $2\theta)\,i$

$=$ expansion of $\{\log(1-xe^{-\alpha i}) - \log(1-xe^{\alpha i})\}$;

$\therefore\ \theta = x\sin\alpha + \frac{1}{2}x^2\sin 2\alpha + \frac{1}{3}x^3\sin 3\alpha + \ldots$

Bernoulli's Numbers.

660. The following is an application of the exponential theorem, which is extensively used in the expansion of functions.

661. *In the expansion of* $(x+1)^n - x^n$, *for integral values of* n *greater than* 1, *substitute* A_r *for* x^r *and equate the result to zero.*

Thus put $n=2$; then $2A_1 + 1 = 0$; $\therefore\ A_1 = -\frac{1}{2}$:

Put $n=3$; then $3A_2 + 3A_1 + 1 = 0$; $\therefore\ A_2 = +\frac{1}{6}$:
and so on.

The substitutive function $(x+1)^n - x^n$ may be simplified by subtracting its values for different values of n; since, after substituting, the result has to be equated to zero.

Thus take $(x+1)^n - x^n$ from $(x+1)^{n+1} - x^{n+1}$;
take the remainder i.e. $x(x+1)^n - x^n(x-1)$
from $x(x+1)^{n+1} - x^{n+1}(x-1)$;
take the remainder i.e. $x^2(x+1)^n - x^n(x-1)^2$
from $x^2(x+1)^{n+1} - x^{n+1}(x-1)^2$;
and so on.

Thus the substitutive function becomes $x^m(x+1)^n - x^n(x-1)^m$.

(1) Here put $m=n$; then we have $x^n(x+1)^n - x^n(x-1)^n$.

Here all the terms involving *even* powers of x cancel; hence the *odd* terms between A_n and A_{2n} alone remain.

Put $n=2$; thus $A_2=0$. And since every increase of n by 1 brings in *one* new odd term, we see that

After A_1 *all the coefficients with odd suffixes vanish.*

(2) Now put $n=m+1$; then we have

$$x^m(x+1)^{m+1}-x^{m+1}(x-1)^m.$$

In this expansion we may omit the *odd* powers of x since the substitutes for them are zero.

The term involving x^{2m-2n} in $x^m(x+1)^{m+1}$ is

$$\frac{(m+1)\,m\,\ldots\,(m-2n+1)}{\lfloor 2n+1}\,x^{2m-2n}.$$

The term involving x^{2m-2n} in $-x^{m+1}(x-1)^m$ is

$$\frac{m\,(m-1)\,\ldots\,(m-2n)}{\lfloor 2n+1}\,x^{2m-2n}.$$

Hence in the substitutive function the term involving x^{2m-2n}

is $$\left(1+\frac{m-2n}{m+1}\right)\times\{\text{term in } x^m(x+1)^{m+1}\},$$

i.e. $$\frac{2m-2n+1}{m+1}\times\{\text{term in } x^m(x+1)^{m+1}\}.$$

Let us write then $C_{2r}=(2r+1)\,A_{2r}$.

Then *the values of* C_{2r} *may be found by substituting* C_{2r} *for* x^{2r} *in the expansion of* $x^m(x+1)^{m+1}$ *equated to zero, the odd coefficients being dropped.*

We have $C_2=3A_2=\frac{1}{2}$.

Put $m=2$; then $3C_4+C_2=0$; $\therefore\ C_4=-\frac{1}{6}$.

Put $m=3$; then $4C_6+4C_4=0$; $\therefore\ C_6=\frac{1}{6}$.

Put $m=4$; then $5C_8+10C_6+C_4=0$; $\therefore\ C_8=-\frac{3}{10}$.

Put $m=5$; then $6C_{10}+20C_8+6C_6=0$; $\therefore\ C_{10}=\frac{5}{6}$.

Put $m=6$; then $7C_{12}+35C_{10}+21C_8+C_6=0$; $\therefore\ C_{12}=-\frac{691}{2730}$.

Put $m=7$; then $8C_{14}+56C_{12}+56C_{10}+8C_8=0$; $\therefore\ C_{14}=\frac{35}{2}$.

And so on.

In the above, write $B_r = (-1)^{r+1} A_{2r}$; then B_1, B_2, B_3... are called **Bernoulli's numbers.**

The relations between the three sets of coefficients here introduced should be carefully noted. Thus

$$B_r = (-1)^{r+1} A_{2r}\,;\quad C_{2r} = (2r+1)\,A_{2r}\,;\quad \therefore\ B_r = \frac{(-1)^{r+1} C_{2r}}{2r+1}.$$

The introduction of the A-coefficients leads to the simplest definition; while that of the C-coefficients simplifies the calculations.

662. *To expand* $\dfrac{x}{e^x - 1}$ *in powers of* x.

[It should be observed that if $e^x = 1+y$; this expression becomes $\dfrac{\log(1+y)}{y}$ which (if $y<1$) can be expanded in powers of y.]

When $x = 0$, the given expression $= 1$; for $e^x = 1 + x + \ldots$

Hence we may write

$$\frac{x}{e^x - 1} = 1 + A_1 x + \frac{A_2 x^2}{\lfloor 2} + \frac{A_3 x^3}{\lfloor 3} + \ldots + \frac{A_n x^n}{\lfloor n} + \ldots$$

Now $e^x - 1 = x + \dfrac{x^2}{\lfloor 2} + \dfrac{x^3}{\lfloor 3} + \ldots + \dfrac{x^n}{\lfloor n} + \ldots$

Hence multiplying and equating the coefficient of x^n (where $n > 1$) to zero; we have

$$\frac{A_{n-1}}{\lfloor n-1 \lfloor 1} + \frac{A_{n-2}}{\lfloor n-2 \lfloor 2} + \frac{A_{n-3}}{\lfloor n-3 \lfloor 3} + \ldots + \frac{A_1}{\lfloor 1 \lfloor n-1} + \frac{1}{\lfloor n} = 0.$$

Multiply by $\lfloor n$. Thus

$$n\,A_{n-1} + \frac{n(n-1)}{\lfloor 2} A_{n-2} + \frac{n(n-1)(n-2)}{\lfloor 3} A_{n-3} + \ldots + n\,A_1 + 1 = 0.$$

That is, A_r may be found by substituting A_r for x^r in the expansions of $(x+1)^n - x^n$ and equating the results to zero.

Hence the coefficients A_1, A_2, A_3... are those found in the last article.

Since $A_1 = -\frac{1}{2}$; and after A_1 the odd coefficients vanish, while $B_r = (-1)^{r+1} A_{2r}$, we may write

$$\frac{x}{e^x - 1} = 1 - \tfrac{1}{2}x + \frac{B_1}{\lfloor 2}x^2 - \frac{B_2}{\lfloor 4}x^4 + \frac{B_3}{\lfloor 6}x^6 - \ldots$$

663. *To derive other expansions from the above.*

We have

$$\frac{x}{e^x - 1} + \frac{x}{2} = \frac{x}{2} \cdot \frac{e^x + 1}{e^x - 1} = \frac{x}{2} \cdot \frac{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}} = \frac{x}{2} \cdot \coth\frac{x}{2},$$

$$\therefore \frac{x}{2}\coth\frac{x}{2} = 1 + B_1\frac{x^2}{\lfloor 2} - B_2\frac{x^4}{\lfloor 4} + B_3\frac{x^6}{\lfloor 6} - \ldots\ldots\ldots\ldots (1).$$

Now $\operatorname{cosech} x = \coth\frac{1}{2}x - \coth x$.

$$\therefore \frac{x}{2}\operatorname{cosech} x = \tfrac{1}{2} - B_1\frac{x^2}{\lfloor 2} + (2^3 - 1)B_2\frac{x^4}{\lfloor 4} - (2^5 - 1)B_3\frac{x^6}{\lfloor 6} - \ldots (2).$$

Again $\frac{1}{2}\tanh\frac{x}{2} = \coth x - \frac{1}{2}\coth\frac{x}{2}$.

$$\therefore \tfrac{1}{2}\tanh\frac{x}{2} = (2^2 - 1)B_1\frac{x}{\lfloor 2} - (2^4 - 1)B_2\frac{x^3}{\lfloor 4} + (2^6 - 1)B_3\frac{x^5}{\lfloor 6} - \ldots (3).$$

Writing here

$$x = iy;\ \cosh\tfrac{1}{2}x = \cos\tfrac{1}{2}y \text{ and } \sinh\tfrac{1}{2}x = i\sin\tfrac{1}{2}y,$$

$$\therefore \frac{y}{2}\cot\frac{y}{2} = 1 - B_1\frac{y^2}{\lfloor 2} - B_2\frac{y^4}{\lfloor 4} - B_3\frac{y^6}{\lfloor 6} - \ldots\ldots\ldots\ldots (4),$$

$$\frac{y}{2}\operatorname{cosec} y = \tfrac{1}{2} + B_1\frac{y^2}{\lfloor 2} + (2^3 - 1)B_2\frac{y^4}{\lfloor 4} + (2^5 - 1)B_3\frac{y^6}{\lfloor 6} + \ldots\ldots (5),$$

$$\tfrac{1}{2}\tan\frac{y}{2} = (2^2 - 1)B_1\frac{y}{\lfloor 2} + (2^4 - 1)B_2\frac{y^3}{\lfloor 4} + (2^6 - 1)B_3\frac{y^5}{\lfloor 6} + \ldots (6).$$

664. Now we have (Art. 526)

$$\cot y = \frac{1}{y} - \Sigma_1^{\infty}\frac{2y}{r^2\pi^2 - y^2},$$

$$\therefore \frac{y}{2}\cot\frac{y}{2} = 1 - 2y^2\Sigma_1^{\infty}\frac{1}{2^2r^2\pi^2}\left(1 - \frac{y^2}{2^2r^2\pi^2}\right)^{-1}.$$

If $y < 2\pi$, *all* the terms can be expanded in ascending powers of y^2 by division. Hence

$$\frac{y}{2}\cot\frac{y}{2} = 1 - 2S, \text{ where}$$

$$S = \frac{y^2}{2^2\pi^2}\left(1 + \frac{y^2}{2^2\pi^2} + \frac{y^4}{2^4\pi^4} + \ldots\right) + \frac{y^2}{4^2\pi^2}\left(1 + \frac{y^2}{4^2\pi^2} + \frac{y^4}{4^4\pi^4} + \ldots\right)$$
$$+ \frac{y^2}{6^2\pi^2}\left(1 + \frac{y^2}{6^2\pi^2} + \frac{y^4}{6^4\pi^4} + \ldots\right) + \ldots$$

Equating coefficients of y^{2r} in the two expansions for $\frac{y}{2}\cot\frac{y}{2}$, we have

$$\frac{B_r}{\lfloor 2r} = \frac{2}{(2\pi)^{2r}}\left[1 + \frac{1}{2^{2r}} + \frac{1}{3^{2r}} + \frac{1}{4^{2r}} + \ldots\right].$$

Hence the limit of the ratio of each term to the preceding in the expansion of $\frac{y}{2}\cot\frac{y}{2}$ is $\left(\frac{y}{2\pi}\right)^2$: so that the series is convergent if $y < 2\pi$.

EXAMPLES XXI.

1. The quantities $1, i, -1, -i, 1, i,$ &c. form a geometric series. Hence deduce the rules of signs in their multiplication.

2. Show that the addition, subtraction, multiplication, and division of complex quantities yields always a complex quantity.

3. If $(a+bi)(c+di)$ is wholly real, a, b, c, and $-d$ must form a proportion; if it is wholly imaginary, a, b, d, c must form a proportion.

4. If the ratio of two complex quantities is wholly imaginary, then the norm of their sum is equal to the sum of their norms.

5. The modulus of the sum of any number of complex numbers is never greater than the sum of their moduli.

6. Show that the square root of $x + yi$ is

$$\pm\left[\sqrt{\{\tfrac{1}{2}\sqrt{(x^2+y^2)}+\tfrac{1}{2}x\}}+i\sqrt{\{\tfrac{1}{2}\sqrt{(x^2+y^2)}-\tfrac{1}{2}x\}}\right]$$

or

$$\pm\left[\sqrt{\{\tfrac{1}{2}\sqrt{(x^2+y^2)}+\tfrac{1}{2}x\}}-i\sqrt{\{\tfrac{1}{2}\sqrt{(x^2+y^2)}-\tfrac{1}{2}x\}}\right],$$

according as y is positive or negative, where the positive values of the square roots are understood.

7. From the above find the fourth roots and the eighth roots of $+1$ and -1.

8. If ω be one of the complex cube roots of $+1$,

$1 + \omega + \omega^2 = 0$; and $x^2 - xy + y^2 = (\omega x + \omega^2 y)(\omega^2 x + \omega y)$.

9. If $\cos 2\alpha + i \sin 2\alpha = A$ and $\cos 2\beta + i \sin 2\beta = B$, then

$$2\cos(\alpha-\beta) = \sqrt{\frac{A}{B}} + \sqrt{\frac{B}{A}} \text{ and } 2\sin(\alpha-\beta) = \sqrt{\frac{A}{B}} - \sqrt{\frac{B}{A}}.$$

10. If C and D are defined in the same way as A and B above,

$$AB + CD = 2\cos(\alpha+\beta-\gamma-\delta).\{\cos(\alpha+\beta+\gamma+\delta) + i\sin(\alpha+\beta+\gamma+\delta)\},$$

$$(A+B)(C+D) = 4\cos(\alpha-\beta)\cos(\gamma-\delta).\{\cos(\alpha+\beta+\gamma+\delta) + i\sin(\alpha+\beta+\gamma+\delta)\}.$$

11. If A, B, C are the angles of a triangle,

$$\cos 3A + \cos 3B + \cos 3C + 3$$
$$= (\cos A + \cos B + \cos C)(\cos A + \cos B + \cos C - 4\cos A \cos B \cos C)$$
$$+ (\sin A + \sin B + \sin C)(\sin A + \sin B + \sin C - 4\sin A \sin B \sin C).$$

12. If we may put $2\cos A = x + \frac{1}{x}$, $2\cos B = y + \frac{1}{y}$ and so on; show that $2\cos(A + B + C + \ldots) = xyz\ldots + \frac{1}{xyz\ldots}$.

13. Find the fifth roots of -1, *first* by Algebra, and *secondly* by Trigonometry.

14. Express $1 + i$; $\sqrt{3} + i$; $1 - i\sqrt{3}$; $-1 - i$; $1 + i + \sqrt{2}$, $i + 2 - \sqrt{3}$ in the form $\rho(\cos\theta + i\sin\theta)$ where ρ is positive.

15. Find the general values of the sixth roots of 1, -1, i and $-i$.

16. Evaluate

$$(3\sqrt{3}+3i\sqrt{3})^{\frac{1}{3}};\ (81\sqrt{3}-81i)^{\frac{1}{4}};\ (32\sqrt{3}+32i)^{\frac{1}{5}}.$$

17. Expand $\cos 7\theta$ and $\sin 7\theta$ in homogeneous terms involving $\cos\theta$ and $\sin\theta$.

18. Expand $64\cos^7\theta$ and $64\sin^7\theta$ in cosines and sines respectively of multiples of θ.

19. $2^8\sin^3\theta\cos^6\theta = 6\sin\theta + 8\sin 3\theta - 3\sin 7\theta - \sin 9\theta$.

20. $2^8\sin^5\theta\cos^4\theta = 6\sin\theta + 4\sin 3\theta - 4\sin 5\theta - \sin 7\theta + \sin 9\theta$.

21. $x^n + x^{-n} - y^n - y^{-n} =$ (if n is even)

$$(x+x^{-1}+y+y^{-1})(x+x^{-1}-y-y^{-1})\times$$

$$P_1^{\frac{1}{2}(n-2)}\left[x^2+x^{-2}-2(x+x^{-1})(y+y^{-1})\cos\frac{2r\pi}{n}+y^2+y^{-2}+4\cos^2\frac{2r\pi}{n}\right].$$

22. $x^n + x^{-n} - y^n - y^{-n} =$ (if n is odd) $(x+x^{-1}-y-y^{-1})\times$

$$P_1^{\frac{1}{2}(n-1)}\left[x^2+x^{-2}-2(x+x^{-1})(y+y^{-1})\cos\frac{2r\pi}{n}+y^2+y^{-2}+4\cos^2\frac{2r\pi}{n}\right].$$

23. $\cosh n\alpha - \cosh n\beta =$ (if n is even)

$$2^{\frac{n}{2}}(\cosh\alpha-\cosh\beta)(\cosh\alpha+\cosh\beta)\times$$

$$P_1^{\frac{1}{2}(n-2)}\left[\cosh 2\alpha - 4\cosh\alpha\cosh\beta\cos\frac{2r\pi}{n}+\cosh 2\beta+2\cos^2\frac{2r\pi}{n}\right].$$

24. $\cosh n\alpha - \cosh n\beta =$ (if n is odd) $2^{\frac{n-1}{2}}(\cosh\alpha-\cosh\beta)\times$

$$P_1^{\frac{1}{2}(n-1)}\left[\cosh 2\alpha - 4\cosh\alpha\cosh\beta\cos\frac{2r\pi}{n}+\cosh 2\beta+2\cos^2\frac{2r\pi}{n}\right].$$

25. If n is a positive integer, and $\tan\theta < 1$,

$$\cos^n\theta\cos n\theta = 1 - \tfrac{1}{2}n(n+1)\tan^2\theta + \tfrac{1}{24}n(n+1)(n+2)(n+3)\tan^4\theta - \ldots$$

$$\cos^n\theta\sin n\theta = n\tan\theta - \tfrac{1}{6}n(n+1)(n+2)\tan^3\theta + \ldots$$

26. $$2^n = 1 + \frac{3n}{4} + \frac{9n^2}{4.8} + \frac{27n(n^2-1)}{4.8.12} + \frac{81n^2(n^2-2^2)}{4.8.12.16} + \ldots$$

27. $\left(\frac{3}{2}\right)^n = 1 + \frac{5n}{12} + \frac{25n^2}{12.24} + \frac{125n(n^2-1)}{12.24.36} + \frac{625n^2(n^2-2^2)}{12.24.36.48} + \ldots$

28. $e^{ny} = 1 + n\sinh y + \frac{n^2}{\lfloor 2}\sinh^2 y + \frac{n(n^2-1)}{\lfloor 3}\sinh^3 y + \frac{n^2(n^2-2^2)}{\lfloor 4}\sinh^4 y + \ldots$

if y lies between $\log_e(\sqrt{2}-1)$ and $\log_e(\sqrt{2}+1)$.

29. Hence

$$y = \sinh y - \frac{1}{2}\cdot\frac{\sinh^3 y}{3} + \frac{1.3}{2.4}\cdot\frac{\sinh^5 y}{5} - \ldots$$

$$\frac{y^2}{2} = \frac{\sinh^2 y}{2} - \frac{2}{3}\cdot\frac{\sinh^4 y}{4} + \frac{2.4}{3.5}\cdot\frac{\sinh^6 y}{6} - \ldots$$

$$\frac{y^3}{6} = \frac{\sinh^3 y}{\lfloor 3} - \frac{1^2+3^2}{\lfloor 5}\sinh^5 y + \frac{1^2+3^2+5^2}{\lfloor 7}\sinh^7 y - \ldots$$

30. $\frac{\theta^3}{6} = \frac{\sin^3\theta}{\lfloor 3} + \frac{1^2+3^2}{\lfloor 5}\sin^5\theta + \frac{1^2+3^2+5^2}{\lfloor 7}\sin^7\theta + \ldots$

31. If $a_{r+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{r}$, then

$$\theta\log\sec\theta = \tfrac{1}{3}a_3\tan^3\theta - \tfrac{1}{5}a_5\tan^5\theta + \tfrac{1}{7}a_7\tan^7\theta - \ldots$$

$$\tfrac{1}{2}(\log\sec\theta)^2 - \tfrac{1}{2}\theta^2 = \tfrac{1}{2}\tan^2\theta - \tfrac{1}{4}a_4\tan^4\theta + \tfrac{1}{6}a_6\tan^6\theta - \ldots$$

32. If $x = \tan\frac{\pi}{4n}(4k+1)$, where k and n are integers,

$$1 - nx - \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots + (-1)^{\frac{1}{2}n(n+1)}x^n = 0.$$

33. If $\cfrac{1}{a+\beta i+}\ \cfrac{1}{a+\beta i+}\ \ldots$ *ad inf.* $= \cos\phi + i\sin\phi$, then

$$\sin\phi = \tfrac{1}{2}\sqrt{(a^2+\beta^2)}.$$

34. Put into complex form:

$\sin(a+bi)$; $\cos(a+bi)$; $\tan(a+bi)$; $\sec(a+bi)$; $\cot(a+bi)$; $\operatorname{cosec}(a+bi)$; $\log_e(a+bi)$; $\log_e(1+x^i)$; $\tan^{-1}(a+bi)$; a^i; i^a; a^{p+qi}; $(a+bi)^i$; i^i.

35. $\log \dfrac{\sin(x+yi)}{\sin(x-yi)} = 2i \tan^{-1}(\cot x \tanh y)$.

36. $\log \dfrac{\cos(x+yi)}{\cos(x-yi)} = 2i \tan^{-1}(\tan x \tanh y)$.

37. $\tan^{-1}(e^{\theta i}) = \frac{1}{2}n\pi + \frac{1}{4}\pi - \frac{1}{2}i \log \tan(\frac{1}{4}\pi - \frac{1}{2}\theta)$.

38. Find $\dfrac{1}{ai}\operatorname{cosec}^{-1}\left(\dfrac{x}{ai}\right)$; and $(a+ia\tan\phi)^{\log(a\sec\phi)-i\phi}$.

39. $\left(\dfrac{1+\sin\phi+i\cos\phi}{1+\sin\phi-i\cos\phi}\right)^n = \cos\left(\dfrac{n\pi}{2}-n\phi\right)+i\sin\left(\dfrac{n\pi}{2}-n\phi\right)$.

40. If $\tan^2\left(\dfrac{\pi}{8}+\dfrac{\phi i}{2}\right) = \alpha+\beta i$,

$$\alpha = \frac{4-e^{2\phi}-e^{-2\phi}}{(e^{\phi}+\sqrt{2})^2+(e^{-\phi}+\sqrt{2})^2};\quad \beta = \frac{2\sqrt{2}(e^{\phi}-e^{-\phi})}{(e^{\phi}+\sqrt{2})^2+(e^{-\phi}+\sqrt{2})^2}.$$

41. If $i^{i^{i^{\cdots}}}$ to an infinite number of exponents be represented $A+Bi$, then $\tan\dfrac{\pi A}{2} = \dfrac{B}{A}$ and $A^2+B^2 = e^{-\pi B}$.

42. $\cos x \cosh x = \Sigma_0^{\infty} \dfrac{x^n}{\lfloor n} 2^{\frac{n}{2}} \cos\dfrac{n\pi}{2}\cos\dfrac{n\pi}{4}$.

Sum the following series (43—50).

43. $\cos\alpha + x\cos(\alpha+\beta) + x^2\cos(\alpha+2\beta)+\ldots$

44. $\sin\alpha + \dfrac{\sin 2\alpha}{\lfloor 2} + \dfrac{\sin 3\alpha}{\lfloor 3} + \ldots$

45. $1 - \dfrac{\cos\alpha\cos\beta}{\lfloor 1} + \dfrac{\cos^2\alpha\cos 2\beta}{\lfloor 2} - \dfrac{\cos^3\alpha\cos 3\beta}{\lfloor 3} + \ldots$

46. $\sin\alpha\cos\beta - \frac{1}{2}\sin^2\alpha\cos 2\beta + \frac{1}{3}\sin^3\alpha\cos 3\beta - \ldots$

47. $x\sin\alpha - \frac{1}{3}x^3\sin 3\alpha + \frac{1}{5}x^5\sin 5\alpha - \ldots$

48. $\sin(\alpha+\beta) - \frac{1}{2}\sin(\alpha+2\beta) + \frac{1}{3}\sin(\alpha+3\beta) - \ldots$

49. $e^x\sin y - \frac{1}{2}e^{2x}\sin 2y + \frac{1}{3}e^{3x}\sin 3y - \ldots$

50. $1+\frac{p}{q}\cos\alpha\sin\beta+\frac{p(p-q)}{\underline{|2}\,.\,9^2}\cos^2\alpha\sin 2\beta$
$+\frac{p(p-q)(p-2q)}{\underline{|3}\,.\,q^3}\cos^3\alpha\sin 3\beta+\ldots.$

Sum the following series to n terms (51—54).

51. $1+\cos\alpha\cos\beta+\cos^2\alpha\cos 2\beta+\cos^3\alpha\cos 3\beta+\ldots$

52. $x\sin\alpha-x^2\sin(\alpha+\beta)+x^3\sin(\alpha+2\beta)-\ldots.$

53. $\sin\alpha+2\sin 2\alpha+3\sin 3\alpha+\ldots.$

54. $1^2\cos\alpha+2^2\cos 2\alpha+3^2\cos 3\alpha+\ldots.$

55. Find

$$1+n\cos\alpha+\frac{n(n-1)}{1\,.\,2}\cos 2\alpha+\frac{n(n-1)(n-2)}{\underline{|3}}\cos 3\alpha+\ldots,$$

where n is a positive integer.

Expand in ascending powers of x the following (56—61).

56. $(1-2x\cos\alpha+x^2)^{-1}(1-x^2)$.

57. $e^{ax}\cos^{bx}$.

58. $e^{x\cos\beta}\cos(\alpha+x\sin\beta)$.

59. $e^{ax}\cos bx+e^{bx}\cos ax$.

60. $\dfrac{\sin\alpha-x\sin(\alpha-\beta)}{1-2x\cos\beta+x^2}$.

61. $\dfrac{1+x\cos\theta}{1+2x\cos\theta+x^2}$.

62. If $\tan x=n\tan y$, then a value of x is

$$y-m\sin 2y+\tfrac{1}{2}m^2\sin 4y-\tfrac{1}{3}m^3\sin 6y+\ldots$$

where $m=\dfrac{1-n}{1+n}$.

63. The side c of a triangle, whose sides a, b $(b<a)$ and angle C are known, is given by

$$\log c=\log a-\frac{b}{a}\cos C-\tfrac{1}{2}\frac{b^2}{a^2}\cos 2C-\tfrac{1}{3}\frac{b^3}{a^3}\cos 3C-\ldots.$$

64. The fraction $\dfrac{ax^2+bx+c}{x^3-1}$ is equal to the sum of three fractions of the form $\dfrac{A}{x-\cos\frac{1}{3}2\pi r-i\sin\frac{1}{3}2\pi r}$, where r has any

three consecutive values: and two of the numerators are imaginary, unless $a = b = c$ in which case both $= 0$.

65. If $\alpha, \beta, \gamma \ldots$ be the n roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n = 0,$$

then, for any values of θ and x,

$$\tan^{-1} \frac{a \sin \theta}{a \cos \theta - x} + \tan^{-1} \frac{\beta \sin \theta}{\beta \cos \theta - x} + \tan^{-1} \frac{\gamma \sin \theta}{\gamma \cos \theta - x} + \ldots$$

$$\equiv \tan^{-1} \frac{p_1 \sin \theta x^{n-1} + p_2 \sin 2\theta x^{n-2} + \ldots + p_n \sin n\theta}{x^n + p_1 \cos \theta x^{n-1} + p_2 \cos 2\theta x^{n-2} + \ldots + p_n \cos n\theta}.$$

66. If p and n be any integers and $\omega_1, \omega_2, \ldots \omega_{n-1}$ be the n^{th} roots of unity, excluding unity itself, then the remainder when p is divided by n is

$$\frac{n-1}{2} + \omega_1^p \frac{\omega_1}{1-\omega_1} + \omega_2^p \frac{\omega_2}{1-\omega_2} + \ldots + \omega_{n-1}^p \frac{\omega_{n-1}}{1-\omega_{n-1}}.$$

67. By making x complex in the expansion of $\sin x$ into factors, show that

$$\tan^{-1} \frac{1}{\pi^2} + \tan^{-1} \frac{1}{4\pi^2} + \tan^{-1} \frac{1}{9\pi^2} + \ldots \text{ ad inf.}$$

$$= \frac{\pi}{4} - \tan^{-1} \left(\cot \frac{1}{\sqrt{2}} \coth \frac{1}{\sqrt{2}} \right).$$

68. From the equation $\log \sec (x + \partial) = \log \sec x + \partial \tan x + \ldots$ show that $\log \sec x = \Sigma \frac{2^{2n}(2^{2n}-1)}{2n \,\underline{|2n}} B_n x^{2n}$.

69. Hence since $\log (2x \operatorname{cosec} 2x) - \log (x \operatorname{cosec} x) = \log \sec x$, show that $\log (x \operatorname{cosec} x) = \Sigma \frac{2^{2n}}{2n \,\underline{|2n}} B_n x^{2n}$.

70. Hence since $\log (\tan x \div x) = \log \sec x - \log (x \operatorname{cosec} x)$, show that $\log \left(\frac{\tan x}{x} \right) = \Sigma \frac{2^{2n}(2^{2n}-2)}{2n \,\underline{|2n}} B_n x^{2n}$.

CHAPTER XXII.

GEOMETRICAL INTERPRETATION OF IMAGINARIES.

VECTORS.

665. IN Chap. XII. the symbol (LM) was used to represent the line LM, regarded as drawn from L to M. It was called a *directed length.*

In that chapter, the operation of adding directed lengths *in the same line* was explained so as to lead to the formula

$$(LM) = (LT) + (TM) \cdot$$

for any points L, M, T *in the same line.*

A directed length could thus be represented by an algebraical number, *affected by a plus or minus sign.*

666. We now extend the use of directed lengths, so as to combine in operation directed lengths in *different* directions.

Here our symbols are *bound down* to a particular *direction* in space as well as to a particular length.

The symbols in Chap. XII. should rather have been called (merely) *affected* lengths than *directed* lengths: but *Vector* is the term used to express directed lengths in the extended sense to be now explained.

667. We may give the following definition:

DEF. The position of any one point in space with respect to any other is called a **Vector.**

Now the position of any point B with respect to any point A is determined by (1) the **length**, (2) the **direction** of the line AB.

The vector involves, therefore, both length and direction.

The symbol (AB) may be extended to represent the vector AB.

668. If the line AB moves always parallel to itself, then its vector remains the same.

In other words, if (AB) and (CD) are equal, parallel, and in the same sense,

$$\text{Vector } (AB) = \text{Vector } (CD).$$

669. The extended use of directed lengths will of course involve the formulæ:

$$\text{Vector } (AB) = -\text{ Vector } (BA).$$
$$\text{Vector } (AA) = 0.$$

Addition of Vectors.

670. In Art. 340, it was shown that if A, B, C be any three points in space, the sum of the projections of (AB) and (BC) upon any line in the plane ABC is equal to the projection of (AC) upon that line.

For this reason it is convenient to define addition of vectors by the equation

$$(AB) + (BC) = (AC).$$

This of course includes the special case when A, B, C are in the same line. In other cases, the equation cannot be directly interpreted arithmetically, but only indirectly: i.e. by projecting upon any line in the plane ABC.

671. The above addition of vectors shows that the sum of two vectors drawn to and from a common point is the vector drawn from the initial point of the first to the final point of the second.

It follows at once that

The vector-sum of the sides of any closed figure, directed from point to point round the figure, is zero.

This is equivalent to Prop. II. Art. 341.

672. If two vectors are expressed by lines drawn from the *same* point, e.g. (AB), (AD); then the sum of (AB) and (AD) is found by drawing (BC) equal, parallel, and in the same sense as (AD), so that we have

$$(AB) + (BC) = (AC),$$
but (Art. 668) $(BC) = (AD)$,
$$\therefore\ (AB) + (AD) = (AC).$$

In other words,

The sum of two vectors, drawn from the same *point, is a vector drawn from that point and represented by the* diagonal of the parallelogram *of which the two given vectors are* adjacent sides.

673. *If three non-coplanar axes be drawn from any point* O, *any vector* (OP) *can be resolved into the sum of three vectors in the direction of these axes.*

For, let OX, OY be any two axes in the plane of the paper: and OZ any axis not in this plane. Then from any point P we may draw PN parallel to OZ, cutting the paper in N. And from N we may draw NM parallel to OY, cutting OX in M. Thus

$$(OP) = (OM) + (MN) + (NP),$$

where (OM) is along OX, (MN) parallel to OY, and (NP) parallel to OZ.

674. In the above result, the *lengths* of OM, MN, NP are entirely determined by the *length* and the *direction* of OP: and *vice versâ*.

That is; *The vector* (OP) *depends on three numbers.*

Hence to equate *one* vector with another involves our equating each of *three* lengths with the corresponding parallel length.

675. If we are confined to a *single plane*, then any vector may be resolved into the sum of *two* vectors in the direction of any *two* axes drawn in the plane from a point.

Hence in this case to equate any *one* vector with another involves our equating each of *two* lengths with the corresponding parallel length.

Division of Vectors.

676. Since equal and parallel lines drawn in the same sense have equal vectors, we may represent any vector by a line drawn from an arbitrarily chosen point O.

Again, since any two straight lines drawn from a point will lie on one plane, we may represent any *two* vectors by two lines (OA), (OB) drawn in the plane of the paper.

677. Draw (OA), (OB) in the plane of the paper to represent any two vectors.

Now a certain operation must be performed upon (OA) in order to change it into (OB).

This operation may be called the *division* of (OB) by (OA): i.e.

$\frac{(OB)}{(OA)}$ **means the operation of changing** (OA) **into** (OB).

678. If (OA) and (OB) were in the *same* line and in the same sense the quotient of (OB) by (OA) would be the ratio of (OB) to (OA): i.e. the number by which the length of OA would have to be multiplied to give the length of OB.

In this particular case, then, the above definition of division would coincide with the ordinary definition.

679. But if (OA) and (OB) were in *different* lines, then we should have

(1) To increase or decrease the length of OA to the length of OB: and then

(2) To revolve the changed length through an angle AOB in the plane AOB.

These two operations may be regarded as *factors* of the total operation, because the second is to be performed upon the result obtained by performing the first.

Also it is clear that the two operations may be performed in either order.

The two factors of the operation have received the following names:

The *stretching* factor is called the *Tensor*:

The *turning* factor is called the *Versor*.

680. It is clear that the Tensor is a mere abstract arithmetic number.

Thus; if the length of OA is 5 feet, and that of OB 3 feet; the *Tensor* of $\frac{(OB)}{(OA)} = \frac{3}{5}$.

In other words the Tensor is *signless*, *directionless*, and *lengthless*.

We may then discuss the *Versor* independently of the *Tensor*.

681. The *Versor* or turning factor requires for its determination a knowledge of (1) the *plane* in which the rotation takes place, (2) the *angle* which measures the *amount* of rotation.

682. If we still take the plane of the paper to represent the plane in which the rotation takes place, we may place this plane in an infinite number of directions.

Now the direction of a plane is assigned by the direction of a perpendicular to it, which is called its *axis*.

This axis is a mere *direction*; not a *vector*.

Just as in ordinary Geometry we abstract from the *direction* of a line and consider only its *length*, so here we abstract from the *length* of the line and consider only its *direction*.

A vector may in fact be defined either as a *directed length* or a *lengthed direction*. And, conversely, a length may be called a *vector divested of direction*; while a direction may be called a *vector divested of length*.

To fix a direction we require to know *two angles*, viz.: (1) the angle which the direction makes with any arbitrarily assumed direction, and (2) the angle which the plane containing the two directions makes with any arbitrarily assumed plane passing through this arbitrarily assumed direction.

683. We thus find that the vector-quotient involves *four* abstract numbers :—

(1) The ratio of the lengths of the two vectors.

(2) The measure of the angle between them.

(3) The measure of the angle between the perpendicular to them and any assumed direction.

(4) The measure of the angle which the plane containing this perpendicular and the assumed direction makes with any assumed plane containing this assumed direction.

For this reason Sir W. R. Hamilton called the vector-quotient a *quaternion.*

684. We may now mainly consider vector-quotients which have the same axis. And this axis we may take to be perpendicular to the plane of the paper.

685. *On the multiplication of operations.*

If, upon the result obtained by performing one operation, we perform a second operation; then the compound operation is called the multiplication of the first by the second.

Thus, if O_1, O_2 are any two operations, O_2O_1, i.e. O_1 multiplied by O_2, means 'performing O_2 upon the result obtained by performing O_1.'

Similarly, O_1O_2, i.e. O_2 multiplied by O_1, means 'performing O_1 upon the result obtained by performing O_2.'

It is clear that such multiplication is not necessarily *commutative:* i.e. O_2O_1 does not necessarily equal O_1O_2. E.g. sin log x does not equal log sin x.

686. *To multiply two versors having the same axis.*

It is clear that the operation of turning a line through an angle θ in any plane and thence through an angle ϕ in that plane is equivalent to turning it through an angle $\theta+\phi$ in that plane.

The multiplication of these two operations is therefore *commutative.*

If OZ is the axis perpendicular to the plane of revolution, the above result suggests that we might represent the operation of turning a line in a plane perpendicular to OZ through an angle θ by the symbol Z^θ.

For we should have the fundamental equation

$$Z^\theta \times Z^\phi = Z^{\theta+\phi}.$$

Here it must be observed that Z is not an *algebraical symbol*, but a symbol of operation. Thus $Z = Z^1$ means 'turning a line through the unit angle in the plane perpendicular to Z.'

[We need not at present define the unit angle.]

687. If (OA) and (OB) be any vectors in the plane perpendicular to OZ; and

(1) If the *arithmetic* ratio of OB to OA is p: and

(2) If the angle AOB, with its *positive or negative* sign is θ,

then the vector-quotient $\dfrac{(OB)}{(OA)}$ may be written $p \,.\, Z^\theta$.

688. *The addition of vector-quotients.*

The addition of vector-quotients is defined so that the *distributive law* shall hold; i.e.

$$(p\,Z^\theta + q\,Y^\phi)\,(OX) \text{ means } p\,Z^\theta\,(OX) + q\,Y^\phi\,(OX),$$

where $p \,.\, Z^\theta$ and $q \,.\, Y^\phi$ are any vector-quotients and (OX) any vector.

689. *To show that the distributive law holds in regard to the multiplication of the sum of two vectors by a vector-quotient in their plane.*

That is, to show that

$$p \,.\, Z^{\theta}\{(OA)+(OB)\} = p \,.\, Z^{\theta}(OA) + p \,.\, Z^{\theta}(OB).$$

If OC is the diagonal of the parallelogram $AOBC$, then

$$(OA)+(OB)=(OC).$$

Thus the left hand of the above equation represents *the vector obtained by multiplying* (OC) *by* p *and turning it through an angle* θ.

But the right hand represents the diagonal of the parallelogram whose sides are (OA) and (OB), *each multiplied by* p *and turned through* θ.

This new parallelogram is clearly similar to the old, but on a different scale and shifted through an angle.

Hence the above equation is proved.

690. Combining the definition of Art. 688 with the theorem of Art. 689, we have the important result that

The multiplication of vector-quotients in the same plane obeys the distributive law.

That is,

$$\begin{aligned} p \,.\, Z^{\theta}(q \,.\, Z^{\phi} + r \,.\, Z^{\psi}) &= pZ^{\theta} qZ^{\phi} + pZ^{\theta} rZ^{\psi} \\ &= pq \,.\, Z^{\theta+\phi} + pr \,.\, Z^{\theta+\psi}. \end{aligned}$$

For, if the operations indicated are performed upon any vector in the plane perpendicular to Z, the above equivalence of operations follows at once from Arts. 688, 689.

691. *To express the versor whose angle is a sub-multiple of four right-angles.*

Since, the operation of turning through four right-angles in any plane leaves a vector unchanged, we have

$$Z^{2\pi} = 1 :$$

and generally $Z^{2r\pi} = 1$, where r is any integer.

Now $Z^{2r\pi} = (Z^{\frac{2r\pi}{n}})^{n}$, where n is any integer.

$\therefore$ $Z^{\frac{2r\pi}{n}}$ is an n^{th} root of 1.

Putting $r = 0,\ 1,\ 2, \ldots n-1$ successively, we see that the versors Z^0, $Z^{2\pi/n}$, $Z^{4\pi/n}$, $Z^{6\pi/n} \ldots Z^{(n-1)2\pi/n}$, which turn the vector through the angles $0,\ 2\pi/n,\ 4\pi/n \ldots (n-1)\,2\pi/n$ respectively are n^{th} roots of 1.

These n versors are clearly all different.

But giving to r higher values, the same versors are repeated.

Hence we have a geometrical interpretation of the algebraical theorems:

(1) There are n and only n n^{th} roots of unity:

(2) The n n^{th} roots are integral powers of some one n^{th} root.

Moreover regarded as multipliers the roots of unity do not affect *magnitude*, but merely *direction*.

692. In the above, let $n = 2$. Then Z^0 and Z^π are the two *square* roots of 1. That is, $Z^0 = 1$, and $Z^\pi = -1$.

That is, *turning a vector through two right-angles is equivalent to changing its sign.* This is the old convention with respect to *affected* lengths.

693. Again, put $n = 4$. Then Z^0, $Z^{\pi/2}$, Z^π, $Z^{3\pi/2}$ are the *fourth* roots of 1.

Moreover $Z^{\pi/2}$ is a square root of Z^π, i.e. $\sqrt{(-1)}$.

Taking then $Z^{\pi/2}$ to be $+i$; $Z^{3\pi/2} = Z^\pi \times Z^{\pi/2} = -i$.

Thus Z^0, $Z^{\pi/2}$, Z^π, $Z^{3\pi/2}$ are respectively $1,\ i,\ -1,\ -i$.

694. The above result gives us an interpretation of i. Thus i means 'the operation of turning a vector in some plane through a right-angle.'

It should be noticed however that this symbol i is incapable of indicating in what plane the rotation is to take place.

We might write i_z to indicate a rotation whose axis is OZ: so that i_z would mean $Z^{\pi/2}$; i_y would mean $Y^{\pi/2}$ and so on. [A different notation was, however, adopted by Hamilton, which need not here be considered.]

695. *To express any vector-quotient as a complex number.*

Let (OH) and (OA) be any two vectors perpendicular to the axis Z.

From H draw HB perpendicular to OA.

Then $(OH) = (OB) + (BH)$.

$\therefore$, by Art. 688, $\dfrac{(OH)}{(OA)} = \dfrac{(OB)}{(OA)} + \dfrac{(BH)}{(OA)}$.

Let the abstract algebraic ratio $\dfrac{OB}{OA}$ be a:

and the abstract algebraic ratio $\dfrac{BH}{OA}$ be b.

Then, since (OB) is in the same line as (OA);

$$\therefore \frac{(OB)}{(OA)} = a.$$

But since (BH) is perpendicular to (OA),

$$\therefore \frac{(BH)}{(OA)} = b.\ Z^{\pi/2} = bi_z.$$

Here a and b are real: i.e. a is positive or negative according as (OB) is along or opposite to (OA): and b is positive or negative according as (BH) makes with (OA) an angle $+\frac{1}{2}\pi$ or $-\frac{1}{2}\pi$.

$$\therefore \frac{(OH)}{(OA)} = a + bi_z.$$

Now the above construction for determining a and b is absolutely unique. Hence any equation such as

$$a + bi_z = c + di_z$$

requires that $a = c$ and $b = d$.

696. Moreover, if θ = angle between (OA) and (OH),

$$OB = OH \cos\theta, \text{ and } BH = OH \sin\theta;$$

$$\therefore a = \frac{OH}{OA}\cos\theta, \text{ and } b = \frac{OH}{OA}\sin\theta.$$

$$\therefore \frac{(OH)}{(OA)} = \frac{OH}{OA} \cdot (\cos\theta + i_z \sin\theta).$$

$$\therefore \text{ versor of } \frac{(OH)}{(OA)} \equiv Z^\theta = \cos\theta + i_z \sin\theta.$$

697. *To find the cosine and sine of the sum of two angles.*

Since $\cos\theta + i_z \sin\theta$ and $\cos\phi + i_z \sin\phi$ denote respectively the operations of turning a vector through angles θ and ϕ in the plane whose axis is Z,

$$\therefore \cos(\theta + \phi) + i_z \sin(\theta + \phi) = (\cos\theta + i_z \sin\theta)(\cos\phi + i_z \sin\phi).$$

But, by Art. 690, we may use the distributive law in the multiplication of these versors. Hence

$$(\cos\theta + i_z \sin\theta)(\cos\phi + i_z \sin\phi)$$
$$= \cos\theta\cos\phi + i_z \sin\theta\cos\phi + i_z \cos\theta\sin\phi - \sin\theta\sin\phi,$$

for $i_z \times i_z$ i.e. $Z^{\pi/2} \times Z^{\pi/2} = Z^\pi = -1$. Hence

$$\cos(\theta + \phi) + i_z \sin(\theta + \phi)$$
$$= \cos\theta\cos\phi - \sin\theta\sin\phi + i_z(\sin\theta\cos\phi + \cos\theta\sin\phi).$$

Hence, by Art. 695,

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi :$$

and

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi.$$

This proof is in principle the same as that given in Art. 343.

698. If in the above we put $\phi = -\theta$, we have

$$(\cos\theta + i_z \sin\theta)\{\cos(-\theta) + i_z \sin(-\theta)\} = \cos(\theta - \theta) + i_z \sin(\theta - \theta).$$

But $\cos(-\theta) = \cos\theta$; $\sin(-\theta) = -\sin\theta$; $\cos 0 = 1$; $\sin 0 = 0$.

$$\therefore \cos^2\theta + \sin^2\theta = 1.$$

This is a proof of Euc. I. 47. For the reader will perceive that this proposition has not been surreptitiously assumed anywhere.

699. From the above result De Moivre's theorem follows at once.

For $\cos\theta + i_z \sin\theta$ denotes the operation of turning through an angle θ.

$\therefore$ $(\cos\theta + i_z \sin\theta)^n$ denotes the operation of turning through an angle θ, n times: i.e. through an angle $n\theta$.

$\therefore$ $(\cos\theta + i_z \sin\theta)^n = \cos n\theta + i_z \sin n\theta$.

It should be pointed out, however, that this equation of operations has no value for the purposes of expanding $\cos n\theta$ and $\sin n\theta$, unless we take into account the *distributive law* proved in Art. 690, by means of which we may 'multiply out' the power on the left-hand according to ordinary algebraic rules. The fundamental proposition upon which the whole of analytical trigonometry rests is the 'addition-theorem' of Art. 697, from which the Demoivrean expansions algebraically follow.

700. Turning back to Art. 695, we have

$$\frac{(OH)}{(OA)} = a + bi_z = \frac{OH}{OA} \cdot (\cos\theta + i_z \sin\theta).$$

But, by Art. 698, $OH^2 = OB^2 + BH^2$.

$$\therefore \frac{OH^2}{OA^2} = \frac{OB^2}{OA^2} + \frac{BH^2}{OA^2} = a^2 + b^2.$$

$\therefore$ the *tensor* of $\dfrac{(OH)}{(OA)} \equiv \dfrac{OH}{OA} = \sqrt{(a^2 + b^2)}$;

and the *versor* of $\dfrac{(OH)}{(OA)} \equiv Z^\theta = \cos\theta + i_z \sin\theta$.

Thus we find

(1) Any vector-quotient is represented by a *complex number*.

(2) Its tensor is the *modulus* of that complex number.

(3) Its versor is the *Demoivrean function* of the angle between the two vectors.

701. Since $p.Z^\theta \times q.Z^\phi = pq.Z^{\theta+\phi}$, we have the fundamentally important theorem that

The modulus of the product *of two complex numbers is equal to the product of their* moduli.

702. *To expand the* cosine *and* sine of an angle in *terms of its* circular measure.

Let (OA) be any initial vector.

Let a moving point trace out the path $(AB)+(BC)+(CD)+\ldots$, in which the angles OAB, OBC, OCD,... are all right-angles, and

$$\frac{AB}{OA}=\frac{BC}{OB}=\frac{CD}{OC}=\ldots=\text{some constant } k, \text{ say.}$$

Then $$\frac{OB}{OA}=\frac{OC}{OB}=\frac{OD}{OC}=\ldots=\sqrt{(1+k^2)}=\rho, \text{ say.}$$

Thus, if M, N are the n^{th} and $(n-1)^{\text{th}}$ points, respectively, of the series A, B, $C\ldots$, then

$$ON=\rho^n.\,OA \text{ and } MN=k\rho^{n-1}.\,OA.$$

Thus the length of the path traced by the moving point from A to N is

$$k\,(1+\rho+\rho^2+\ldots+\rho^{n-1}).\,OA=\frac{k\,(\rho^n-1)}{\rho-1}.\,OA.$$

Now, let $k=\theta/n$: and let n increase indefinitely, while θ remains finite, so that k diminishes indefinitely.

Then it will be seen that limit of $\rho^n=1$; and limit of

$$\frac{k\,(\rho^n-1)}{\rho-1}=\theta.$$

Hence the tracing point describes a circle of radius OA, and

$$\text{arc } AN=\theta.\,(\text{radius } OA) \quad \ldots\ldots\ldots\ldots\ldots \quad (1).$$

Now the operation to be performed upon each of the vectors (OA), (OB), $(OC)\ldots$ to obtain the next is $1+ki$.

Hence $$\frac{(ON)}{(OA)}=(1+ki)^n=\left(1+\frac{\theta i}{n}\right)^n.$$

Increasing n indefinitely and expanding by the binomial theorem, we have

$$\frac{(ON)}{(OA)}=1+\theta i+\frac{\theta^2 i^2}{\lfloor 2}+\frac{\theta^3 i^3}{\lfloor 3}+\ldots+\frac{\theta^r i^r}{\lfloor r}+\ldots$$

$$=\text{expansional value of } e^{\theta i}.$$

But θ is the circular measure of the angle AON by (1), and $ON = OA$:

$$\therefore \frac{(ON)}{(OA)} = \cos\theta + i\sin\theta.$$

$\therefore \cos\theta + i\sin\theta =$ expansional value of $e^{\theta i}$.

703. In the above, we have used i for the operator which turns a vector through a right-angle in *any* fixed plane.

Let Z denote the operation of turning an angle through some undetermined unit of angular measurement in the plane whose axis is Z. And let A be the measure of θ in terms of this unit. Then

$$Z^A \equiv \cos A + i_z \sin A \equiv \cos\theta + i_z \sin\theta \equiv e^{\theta i_z}.$$

The most convenient unit to take will be the *right-angle*. Then

$$A \text{ right-angles} = \tfrac{1}{2}\pi A \text{ radians: or } \theta = \tfrac{1}{2}\pi A.$$

$$\therefore Z^1 \equiv e^{\frac{1}{2}\pi i_z} \equiv \cos\tfrac{1}{2}\pi + i_z \sin\tfrac{1}{2}\pi = i_z.$$

704. If X, Y, Z be three axes at right-angles to one another, the operations of turning through a right-angle round these axes will be respectively i_x, i_y, i_z. For these three operations Hamilton used the symbols i, j, k.

705. If we have a *fixed plane* for rotation, and a *fixed direction* in that plane for our initial vector, then the operation i upon this initial vector will give a vector *fixed in the plane of rotation* at right-angles to the initial vector.

Thus, since $$\frac{(OH)}{(OA)} = a + bi,$$

where a and b are abstract ratios, having therefore neither length nor direction, we have

$$(OH) = a\,.\,(OA) + bi\,(OA).$$

Now, if $a\,.\,(OA)$ and $b\,.\,(OA)$ be represented by α and β, respectively, α and β will stand for *lengths* having an arbitrarily fixed direction. Thus, since

$$(OH) = \alpha + \beta i,$$

$\therefore$ any vector in the fixed plane is expressed in terms of its projection and erection upon any fixed line in that plane.

In this way, i would come to be associated with the direction which the rotation *produces*, instead of with that of the *axis* which produces the rotation.

706. The student should very carefully compare the equations

$$\frac{(OH)}{(OA)} = a + bi \text{ and } (OH) = a + \beta i.$$

In the first a and b are mere algebraical quantities; and as such are incapable of containing within them either *length* or *direction*. But if these quantities be applied to geometry, we must assign an arbitrary (or unit) length and an arbitrary (or standard) direction. Taking (OA) as our unit and standard, we may write $a.(OA) = \alpha$, and $b(OA) = \beta$. Here α and β are not algebraical quantities at all, but *vectors*.

707. The method of interpreting i explained in the last two articles was historically prior to the method upon which this chapter has been based. Wallis, Buée, Warren, Argand are the chief mathematicians whose names have been associated with this interpretation. Hamilton's grand invention of quaternions may be said to be the immediate result of associating the direction of i with the *axis of rotation*, instead of with the direction *produced by the rotation*.

The following article may be read by the student who wishes to connect the study of Quaternions with the interpretation of i as explained in this chapter.

708. Let X, Y, Z be three axes perpendicular to one another. Suppose them to be of indeterminate length, but having a determinately *positive* sense. Let them be so placed that the rotation from X to Y is left-handed for a person looking at the plane of XY from the positive side of Z. Then it will be found that the rotation from Y to Z is also left-handed to a person looking at the plane YZ from the positive side of X; and the rotation from Z to X is left-handed to a person looking at the plane ZX from the positive side of Y.

We take such left-handed rotation to be *positive*.

Thus, using the notation explained, we have the following equations:—

(1)	$i_z X = Y$;	(6)	$i_x Z = -Y$;
(2)	$i_x Y = Z$;	(5)	$i_y X = -Z$;
(3)	$i_y Z = X$;	(4)	$i_z Y = -X$.

Substituting for Y from (1) in (2), we have

$$i_x i_z X = Z: \text{ but from (5) } i_y X = -Z.$$

$$\therefore i_x i_z = -i_y,$$

if these operations are performed upon X.

This equation should be compared with (6).

By the same process we shall obtain the following six equations of operation:

(1)	$i_z i_x = i_y$;	(6)	$i_x i_z = -i_y$;
(2)	$i_x i_y = i_z$;	(5)	$i_y i_x = -i_z$;
(3)	$i_y i_z = i_x$;	(4)	$i_z i_y = -i_x$.

This second set of equations is immediately obtainable from the first by substituting i_x for X; i_y for Y; i_z for Z.

In (1), (2), (3) the circular order x, y, z, x is maintained; in (4), (5), (6) this circular order is reversed.

It should be noticed that to prove the second set the operand to be chosen must be that which corresponds to the operator which stands first.

The equivalence in form between the first and second set of equations led Hamilton to use the *same* symbol to denote two things; viz.:—

(1) An undetermined (or unit) vector in any direction.

(2) The operation of turning a line about this vector as axis through a right-angle.

It is essential to observe that the multiplication of the operators i_x, i_y, i_z is *not commutative*.

Thus $i_x i_y = -i_y i_x$; $i_y i_z = -i_z i_y$; $i_z i_x = -i_x i_z$.

MISCELLANEOUS EXAMPLES.

1. Show from the definitions of the trigonometrical ratios which apply to angles of any magnitude, that a knowledge of the *tangent* of *half* an angle determines all the trigonometrical ratios of the angle without any ambiguity.

Also write down the value of each ratio of an angle in terms of the tangent of the half-angle in order to illustrate the above.

2. If A', B', C' be the external angles of a triangle ABC,

bc vers $A' + ca$ vers $B' + ab$ vers $C' = \frac{1}{2}(a+b+c)^2$.

3. In any triangle ABC,

$$a^2 \cos 2(B-C) = b^2 \cos 2B + c^2 \cos 2C + 2bc \cos (B-C).$$

4. Show that

$$\tfrac{1}{4}\pi = 2\tan^{-1}\tfrac{1}{8} + \tan^{-1}\tfrac{1}{7} + 2\tan^{-1}\tfrac{1}{5};$$
$$\sin^{-1}\tfrac{3}{5} = \cos^{-1}\tfrac{6}{65} + 2\tan^{-1}\tfrac{1}{5}.$$

5. If

$$\cos A \cos X + \cos B \cos Y + \cos C \cos Z = \cos^2 A + \cos^2 B + \cos^2 C = \cos^2 X + \cos^2 Y + \cos^2 Z = 1,$$

then $$\cos A \sec X = \cos B \sec Y = \cos C \sec Z = 1.$$

6. Eliminate θ from the equations

$$\frac{x}{\sin 3\theta - \sin\theta} = \frac{y}{\cos 3\theta - \cos\theta} = a.$$

7. Eliminate ϕ and ϕ' from

$$x\cos\phi + y\sin\phi = x\cos\phi' + y\sin\phi' = 2a:$$

and $$2\cos\tfrac{1}{2}\phi \cos\tfrac{1}{2}\phi' = 1.$$

8. Eliminate θ from

$$x\tan\theta = y\tan\phi$$

and $$(x+y)\cos(\theta-\phi) + (x-y)\cos(\theta+\phi) = 2a.$$

9. If $a \sin \alpha = b \sin \theta$; and $a \sin \beta = b \sin \phi$; and $\alpha - \beta = \theta - \phi$; and a and b are unequal; then

$$\alpha = \beta + \lambda\pi \text{ and } \theta = \phi + \lambda\pi$$

where λ is any integer.

10. If $$x \cos \theta - y \sin \theta = 2a \cos 2\theta,$$

and $$x \sin \theta + y \cos \theta = 2a \sin 2\theta,$$

then $$(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$$

11. Eliminate θ from

$$b \cos^2 \theta + a \sin^2 \theta = b \cos^2 (\theta + \phi) + a \sin^2 (\theta + \phi) = 0.$$

12. Eliminate θ and ϕ from

$$ax \sec \phi - by \operatorname{cosec} \phi = ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2;$$

and $$\theta - \phi = \tfrac{1}{2}\pi.$$

13. Prove that the equations

$$(x + x^{-1}) \sin \alpha = yz^{-1} + zy^{-1} + \cos^2 \alpha,$$
$$(y + y^{-1}) \sin \alpha = xz^{-1} + zx^{-1} + \cos^2 \alpha,$$
$$(z + z^{-1}) \sin \alpha = xy^{-1} + yx^{-1} + \cos^2 \alpha,$$

are not independent, but are equivalent to

$$x + y + z = x^{-1} + y^{-1} + z^{-1} = -\sin \alpha.$$

14. Solve for α and β

$$\sin \alpha \cosh \beta = \tfrac{3}{4} \text{ and } \cos \alpha \sinh \beta = \tfrac{1}{4}.$$

15. If α and β are the two values of $\sin \theta$ which satisfy the equation

$$a \cos 2\theta + b \sin 2\theta = c,$$

prove that $$\alpha^2 + \beta^2 = \frac{a^2 + b^2 - ac}{a^2 + b^2}.$$

16. If $$a \cos (\theta + \phi) + b \cos (\theta - \phi) + c = 0,$$
$$a \cos (\phi + \psi) + b \cos (\phi - \psi) + c = 0,$$
$$a \cos (\psi + \theta) + b \cos (\psi - \theta) + c = 0,$$

and if θ, ϕ, ψ are all unequal, then $a^2 - b^2 + 2bc = 0$.

17. Eliminate θ from

$$\cos^3\theta = m\cos(3\theta - \alpha) \text{ and } \sin^3\theta = -m\sin(3\theta - \alpha),$$

showing that $1 + m\cos\alpha = 2m^2$.

18. If α, β, γ be unequal and each less than 2π, and if $\cos(\alpha+\theta)\sec 2\alpha = \cos(\beta+\theta)\sec 2\beta = \cos(\gamma+\theta)\sec 2\gamma$, then $\cos(\beta+\gamma) + \cos(\gamma+\alpha) + \cos(\alpha+\beta) = 0$.

19. Given
$$x\cos\alpha + y\sin\alpha + z + \cos 2\alpha = 0,$$
$$x\cos\beta + y\sin\beta + z + \cos 2\beta = 0,$$
$$x\cos\gamma + y\sin\gamma + z + \cos 2\gamma = 0,$$

then $x\cos\phi + y\sin\phi + z + \cos 2\phi =$

$$8\sin\tfrac{1}{2}(\alpha+\beta+\gamma+\phi)\sin\tfrac{1}{2}(\phi-\alpha)\sin\tfrac{1}{2}(\phi-\beta)\sin\tfrac{1}{2}(\phi-\gamma).$$

20. If α, β, γ be unequal and each less than π, then the system of equations

$$\frac{\sin(2\alpha-\beta-\gamma)}{\cos(2\alpha+\beta+\gamma)} = \frac{\sin(2\beta-\gamma-\alpha)}{\cos(2\beta+\gamma+\alpha)} = \frac{\sin(2\gamma-\alpha-\beta)}{\cos(2\gamma+\alpha+\beta)},$$

is equivalent to

$$\cos 2(\beta+\gamma) + \cos 2(\gamma+\alpha) + \cos 2(\alpha+\beta) = 0.$$

21. If $A + B + C$ lie between 0 and 2π,

$$\left(\frac{1+\cos A-\cos B-\cos C}{\sin\frac{1}{2}B\sin\frac{1}{2}C}\right)^2 + \left(\frac{1+\cos B-\cos C-\cos A}{\sin\frac{1}{2}C\sin\frac{1}{2}A}\right)^2 + \left(\frac{1+\cos C-\cos A-\cos B}{\sin\frac{1}{2}A\sin\frac{1}{2}B}\right)^2$$
$$+\frac{(1+\cos A-\cos B-\cos C)(1+\cos B-\cos C-\cos A)(1+\cos C-\cos A-\cos B)}{2\sin^2\frac{1}{2}A\sin^2\frac{1}{2}B\sin^2\frac{1}{2}C} = 16.$$

22. The area of $ABC = s^2 \cdot 2^{\frac{3}{2}}\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\phi\cos\tfrac{1}{2}\psi$,
where $\cos\theta = \tan\frac{1}{2}B + \tan\frac{1}{2}C$; $\cos\phi = \tan\frac{1}{2}C + \tan\frac{1}{2}A$;
$\cos\psi = \tan\frac{1}{2}A + \tan\frac{1}{2}B$.

23. If L, M, N are the feet of the perpendiculars from A, B, C on the opposite sides, and p_1, p_2, p_3, $-p_4$ the perpen-

diculars from A, B, C, and the ortho-centre on the sides of L, M, N, then p_1, p_2, p_3, p_4 are the roots of

$$x^4 + (\Delta^2/R^2 - 2Rr' - r'^2)\, x^2 = 2Rx^3 + r'^2 \Delta^2/R^2,$$

where r' is the radius of the inscribed circle of LMN.

24. The sides taken in order of a pentagon circumscribed to a circle are a, b, c, d, e: prove that its area is a root of the equation

$$x^4 - x^3 s \{\tfrac{1}{2}\Sigma a^2 (b + e - c - d) - \tfrac{1}{4}\Sigma a^3 + \tfrac{1}{2}\Sigma acd\}$$
$$+ (s-a-c)(s-b-d)(s-c-e)(s-d-a)(s-e-b)\, s^3 = 0.$$

25. The roots of $e^x = 1$ being $2n\pi i$, show that

$$e^x - 1 \equiv \left(1 + \frac{x}{2} + \frac{x^2}{2.4} + \frac{x^3}{2.4.6} + \frac{x^4}{2.4.6.8} + \dots\right)$$
$$\times \left(1 + \frac{x^2}{2^2\pi^2}\right)\left(1 + \frac{x^2}{4^2\pi^2}\right)\left(1 + \frac{x^2}{6^2\pi^2}\right)\left(1 + \frac{x^2}{8^2\pi^2}\right)\dots$$

26. Show geometrically and algebraically that the sum of the operations $r(\cos\alpha + i\sin\alpha)$ and $r'(\cos\beta + i\sin\beta)$ (which have the same axis) is the operation $\rho(\cos\theta + i\sin\theta)$, where

$$\rho^2 = r^2 + r'^2 - 2rr'\cos(\alpha - \beta) \text{ and } \tan\theta = \frac{r\cos\alpha + r'\cos\beta}{r\sin\alpha + r'\sin\beta}.$$

27. From the proposition that the diagonals of a rhombus bisect one another at right-angles, show that $2\cos A$ and $2i\sin A$ can be expressed as the sum and difference, respectively, of z^θ and $z^{-\theta}$ where θ is proportional to A; and z is some undetermined quantity.

28. If $a^2 + b^2 + c^2 + d^2$ be called the norm of the operation $a + bi_x + ci_y + di_z$, show, by means of the formulae of Art. 708, that the norm of the product of two such operations is equal to the product of their norms.

ANSWERS TO EXAMPLES.

EXAMPLES I. pp. 15—17.

1. (1) 3″. (2) 25′. (3) 4^s. (4) $35^s\ 60'$. (5) $188^s\ 80'$.
(6) $23425^s\ 78'\ 90''$. (7) $5204^s\ 6'\ 50'' \cdot 42003$.

2. (1) $0'' \cdot 972$. (2) $13'\ 30''$. (3) $3°\ 36'$. (4) $32°\ 2'\ 24''$.
(5) $169°\ 55'\ 12''$. (6) $21083°\ 12'\ 36'' \cdot 36$. (7) $4683°\ 39'\ 30'' \cdot 73608972$.

3. (1) $\cdot 000004$. (2) $\cdot 007$. (3) $\cdot 0004$. (4) $\cdot 050245$.
(5) $\cdot 26463562$. (6) $\cdot 0001$. (7) $\cdot 00001$. (8) $\cdot 000001$.
(9) $\cdot 123$. (10) $\cdot 345$. (11) $\cdot 678$.

4. (1) $1^s\ 26'\ 19\frac{41}{61}''$. (2) $3^s\ 40'\ 96\frac{3}{61}''$. (3) $67^s\ 80'$.

5. (1) 3°. (2) $40°\ 32'\ 42'' \cdot 972$. (3) $6°\ 22'\ 44'' \cdot 796$.

9. 45°, 60°, 75°. **10.** 36°. **11.** 90°, 81°, 9°.

13. 1170°, 7080°, 3300°. **14.** $(8n+1)45°$, $(8n+3)45°$.

15. (1) 60°. (2) 120°. (3) 108°. (4) 150°. (5) 156°.

16. (1) 5 ft. (2) 5 ft. 5 in. (3) 6 ft. 1 in.

17. (1) 2 ft. (2) 3 ft. 9 in. (3) 6 ft. 5 in.

18. a^2+b^2. **19.** If $OA=1$, then $OB=\sqrt{2}$, $OC=\sqrt{3}$, $OD=\sqrt{4}$ &c.

EXAMPLES II. pp. 37, 38.

1. (1) 22 yds. to within about 3 in. (2) 3 m. 1 fur. 29 yds. 7 in. to within about 3 in. (3) 55 yds. to within about 6 in.

2. (1) $\frac{1}{180}\pi^c$ or $0^c \cdot 017453$. (2) $\frac{1}{200}\pi^c$ or $0^c \cdot 015708$.
(3) $\frac{1}{180}(\pi^2)^c$ or $0^c \cdot 054783$. (4) $\frac{1}{200}(\pi^2)^c$ or $0^c \cdot 049348$.
(5) $\frac{11}{3600}\pi^c$ or $0^c \cdot 035779$. (6) $\frac{31}{3600}\pi^c$ or $0^c \cdot 270522$.

3. (1) 180°. (2) 540°. (3) 450°. (4) $19°\ 5'\ 54''$.
(5) $171°\ 53'\ 31''$. (6) $180°\ 4'\ 21''$.

4. Taking $\pi=\frac{22}{7}$, (1) $5°\ 50'$. (2) 70°. (3) $11°\ 40'$. (4) 210°.

5. 162°, 90°. **6.** (1) 1·0472 ft. (2) 1·7104 ft.
(3) 2·4435 ft. (4) ·7854 ft. **7.** 4 sq. yds. 2 sq. ft. 70 sq. in.

8. $118\frac{8}{4}$ sq. in. **10.** (1) $1 \cdot 2^c$. (2) $2 \cdot 3^c$. **11.** $\frac{1}{6}$, $\frac{1}{14}$, $\frac{1}{4}$.

12. $\frac{4}{15}\pi$, $\frac{8}{15}\pi$, $\frac{2}{3}\pi$, $\frac{12}{15}\pi$, $\frac{14}{15}\pi$.

EXAMPLES III. pp. 52—54.

1. $\sin A = \cos B = \frac{1}{6}\sqrt{11}$; $\cos A = \sin B = \frac{5}{6}$ &c.

2. $\sin A = \cos B = \frac{3}{5}$; $\cos A = \sin B = \frac{4}{5}$; $\tan A = \cot B = \frac{3}{4}$.

3. $\sec A = \operatorname{cox} B = \frac{7}{4}$; $\operatorname{cox} A = \sec B = \frac{7}{15}\sqrt{5}$; $\cot A = \frac{2}{15}\sqrt{5}$; $\cot B = \frac{2}{7}\sqrt{5}$.

9. 20°. **12.** (i) $\cos^4\theta - 4\cos^2\theta + 4$, (ii) $8 - 12\sec^2\theta + 6\sec^4\theta - \sec^6\theta$.

46. (i) $\pm\frac{1}{2}\sqrt{3}$. (ii) 1 or 0. (iii) $\frac{3}{5}$. (iv) $\frac{5}{7}$. (v) $\frac{1}{2}\sqrt{3}$ or $\frac{1}{2}$.

47. $\frac{12}{13}$. **48.** $\frac{3}{4}$ or $\frac{4}{3}$.

EXAMPLES IV. pp. 71, 72.

1. (1) $\cos A = \frac{12}{13}$, $\tan A = \frac{5}{12}$. (2) $\sin A = \dfrac{3}{\sqrt{10}}$, $\cos A = \dfrac{1}{\sqrt{10}}$.

(3) $\sin A = \frac{7}{7}\sqrt{6}$, $\cos A = \frac{5}{7}$. (4) $\sin A = \frac{4}{5}$, $\tan A = \frac{4}{3}$.

(5) $\sin A = \dfrac{3}{\sqrt{10}}$, $\cos A = \dfrac{1}{\sqrt{10}}$. (6) $\sin A = \frac{7}{25}$, $\cos A = \frac{24}{25}$.

2. $\tan A = \dfrac{\sin A}{\sqrt{(1-\sin^2 A)}} = \dfrac{\sqrt{(1-\cos^2 A)}}{\cos A}$.

3. $\sec A = \dfrac{1}{\sqrt{(1-\sin^2 A)}} = \dfrac{\sqrt{(\cot^2 A + 1)}}{\cot A}$.

4. $\operatorname{cox} A = \dfrac{\sec A}{\sqrt{(\sec^2 A - 1)}} = \dfrac{1}{\sqrt{(1-\cos^2 A)}}$.

5. $\sin A = \dfrac{\tan A}{\sqrt{(1+\tan^2 A)}} = \dfrac{\sqrt{(\sec^2 A - 1)}}{\sec A}$.

8. 60°. **9.** 30°. **10.** 45°. **11.** 45°. **12.** 60° or 45°.

13. 0 or 60°. **14.** 45°. **15.** 45° or 90°. **16.** 60°.

17. 15°. **18.** 30°. **19.** 30°. **20.** 18°. **21.** 45° or $67\frac{1}{2}$°.

22. $67\frac{1}{2}$°. **23.** 45° or 30°. **24.** 0 or 60°. **25.** 15° or 75°.

33. Between 45° and 60°. **34.** Between 30° and 60°.

35. $1 - \cos\theta$ increases from 0 to 1; $\sec\theta - 1$ increases from 0 to ∞.
$\sin\theta + \cos\theta$ increases from 1 to $\sqrt{2}$ at 45°, and then decreases to 1 again;
$\sec\theta - \tan\theta$ decreases from 1 to 0.

36. $\alpha = 45°$, $\beta = 60°$. **37.** $A = 37\frac{1}{2}$°, $B = 7\frac{1}{2}$°.

EXAMPLES V. pp. 83—86.

2. $\frac{1}{4}(\sqrt{6}-\sqrt{2})$, $\frac{1}{4}(\sqrt{6}+\sqrt{2})$. **3.** $\frac{1}{4}(\sqrt{6}+\sqrt{2})$, $\frac{1}{4}(\sqrt{6}-\sqrt{2})$.

5. 1, 0. **6.** $\frac{1}{3}\frac{6}{4}\frac{0}{5}$, $\frac{3}{3}\frac{0}{4}\frac{1}{5}\frac{3}{5}$. **7.** $\dfrac{1+\tan A}{1-\tan A}$, $\dfrac{\cot A + 1}{\cot A - 1}$.

17. $2\sin 9A\cos 2A$. **18.** $2\sin 2A\cos 11A$. **19.** $2\cos 4A\cos A$.

20. $2\sin 5A\sin 12A$. **21.** $2\sin nA\cos A$.

22. $2\sin 45°\cos(2A - 45°)$. **23.** $2\sin\frac{1}{2}A\cos\frac{5}{2}A$.

24. $2\sin\frac{1}{2}A\sin\frac{5}{2}A$. **25.** $2\sin(\frac{1}{2}\theta - \frac{1}{2}\phi + 45°)\cos(\frac{1}{2}\theta + \frac{1}{2}\phi - 45°)$.

26. $2\cos(45°-\frac{1}{2}\phi+\frac{1}{2}\theta)\sin(45°-\frac{1}{2}\phi-\frac{1}{2}\theta)$. **27.** $2\sin nA\sin 3A$.
28. $2\cos(A-B+45°)\sin(A+B-45°)$. **29.** $\sin(A+B)+\sin(A-B)$.
30. $\cos(A-B)-\cos(A+B)$. **31.** $\sin 60°-\sin 40°$.
32. $\sin 60°+\sin 40°$. **33.** $\sin 4\theta-\sin\theta$.
34. $\cos B-\cos 6B$. **35.** $\sin 2\alpha+\sin 2\beta$.
36. $\sin 18°-\sin 4°$. **59.** $\tan(A+B)=1\frac{8}{15}$, $\tan(A-B)=\frac{5}{27}$.
68. 45° or 0. **69.** 30°. **70.** 0. **71.** 0. **72.** 45° or 60°.
73. 30° or 90°. **74.** 20°. **75.** 10°. **76.** 18°.

Examples VI. pp. 90, 91.

1. $\sin 135°=\dfrac{1}{\sqrt{2}}$, $\cos 135°=-\dfrac{1}{\sqrt{2}}$, $\tan 135°=-1$.

$\sin 120°=\dfrac{\sqrt{3}}{2}$, $\cos 120°=-\frac{1}{2}$, $\tan 120°=-\sqrt{3}$.

$\sin 150°=\frac{1}{2}$, $\cos 150°=-\dfrac{\sqrt{3}}{2}$, $\tan 150°=-\dfrac{1}{\sqrt{3}}$.

$\sin 112\frac{1}{2}°=\frac{1}{2}\sqrt{(2+\sqrt{2})}$, $\cos 112\frac{1}{2}°=-\frac{1}{2}\sqrt{(2-\sqrt{2})}$, $\tan 112\frac{1}{2}°=-\sqrt{2}-1$.
$\sin 105°=\frac{1}{4}(\sqrt{6}+\sqrt{2})$, $\cos 105°=-\frac{1}{4}(\sqrt{6}-\sqrt{2})$, $\tan 105°=-2-\sqrt{3}$.

9. The relations of inequality hold *numerically*, not algebraically.

Examples VII. pp. 111, 112.

1. 13 ft. **2.** 60°. **3.** $B=45°$ or $\tan B=\frac{3}{11}$.
4. $6\frac{1}{2}$, $2\frac{4}{5}\sqrt{2}$, $3\frac{1}{10}\sqrt{2}$. **5.** $(2\sqrt{3}\pm\sqrt{5})$ ft. **6.** 45°, 60°, 75°.
7. 135°, 30°, 15°. **8.** $1\frac{1}{4}$, 6. **9.** 20° 36′ 35″, 69° 23′ 25″, 90°.
10. 54° and 108°, or 126° and 36°. **11.** 45°. **12.** 7 : 9 : 11.
13. $3-\sqrt{3}$, $37\frac{1}{2}$ sq. ft. **14.** 8. **15.** 4 m. or $2\sqrt{13}$ m.
16. $a=b$, $B=30°$, $C=120°$. **17.** 60°. **19.** $\frac{88}{85}$, $\frac{10}{12}\frac{88}{85}\frac{4}{65}$, $\frac{18}{19}\frac{8}{3}$.
21. $B=60°$, $a=(3+\sqrt{3})$ ft., $b=3\sqrt{2}$ ft., $c=2\sqrt{3}$ ft. **22.** 6 in.

Examples VIII. pp. 117—121.

1. 500 ft. nearly. **2.** 1000 ft. nearly. **3.** 2 ft. 3 in.
4. 18·3 ft. **5.** $\frac{1}{2}\sqrt{2}$ m., $\sqrt{2}$ m. **6.** $5\frac{1}{2}$ minutes, $1\frac{1}{8}$ miles, nearly.
7. 380 ft., 400 ft.
13. $\dfrac{5b^2+8ab-5a^2}{4b(a+b)}$, $\dfrac{3b^2+8ab+13a^2}{12a(a+b)}$, $\dfrac{3b^2+8ab-5a^2}{6ab}$.
18. $25\sqrt{3}$ yds. **23.** $h=c\cdot\dfrac{\cot\beta\pm2\sqrt{(2\cot^2\alpha-\cot^2\beta)}}{2(\cot^2\beta-\cot^2\alpha)}$.

EXAMPLES X. pp. 168—180.

72. (1) $\frac{1}{2}(B+C), \frac{1}{2}(C+A), \frac{1}{2}(A+B)$. (2) $A+\frac{1}{2}B+\frac{1}{2}C, \frac{1}{2}B, \frac{1}{2}C$.
(3) $B+C-A,\ C+A-B,\ A+B-C$.

125. 3; $4\sqrt{(2-\sqrt{2})}=3{\cdot}0614$; $\frac{5}{2}(\sqrt{5}-1)=3{\cdot}0901$; $3\sqrt{2}(\sqrt{3}-1)=3{\cdot}1058$;
$5\{\sqrt{(3+\sqrt{5})}-\sqrt{(5-\sqrt{5})}\}=3{\cdot}1287$.

EXAMPLES XI. pp. 209—215.

2. $1, \frac{1}{a^5}, \sqrt{a^3}, \frac{1}{\sqrt[3]{a^4}}$; $0, -5, \frac{3}{2}, -\frac{1}{4}$. **3.** $\frac{1}{81}$, 81, $\frac{1}{32}$, 1.

5. $x^{\frac{3}{5}}, x^{-7}, x^{-\frac{3}{2}}, x^0, x^{-1}, x^{\frac{23}{6}}, x^{\frac{17}{6}}$; $\frac{3}{5}, -7, -\frac{3}{2}, 0, -1, 2\frac{3}{6}, 1\frac{7}{6}$.

6. $5, 3, 3, 4, 4, 1\frac{7}{6}, \frac{3}{5}, -\frac{3}{5}, \frac{3}{2}, -2, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{5}, -\frac{1}{10}, -\frac{3}{2}, -\frac{7}{6}, -5, -1, -\frac{1}{2}, -1, -2, \frac{7}{5}$.

7. $3, 2, 0, -1, -3, 4, 4, 3, 3, 2, 11$.

24. ·6020600, ·9030900, 2·1072100, ·1003433, 3·010300, $\bar{1}$·6989700, $\bar{1}$·6387640, $\bar{1}$·3979400, $\bar{2}$·494850, 2·3010300, $\bar{3}$.3010300.

25. ·6989700, 1·3979400, 1·6989700, $\bar{1}$·3010300, $\bar{3}$·9030900, $\bar{1}$·4757725, 3·6989700, $\bar{4}$·6989700.

26. ·9542426, $\bar{1}$·5228787, 1·4771213, 2·3856065, $\bar{2}$·4771213, $\bar{1}$·0457574.

27. ·7781513, 1·2552726, 1·1760913, 2·1303339, 2·1583626, 2·8750613, $\bar{1}$·3467874, $\bar{3}$·6478174.

28. 2·5352940, 5·5352940, $\bar{1}$·5352940, $\bar{3}$·5352940, ·5352940.

29. 1·5440680, 10^{-7}; 2·3222193, 10^{-7}; 2·3891660, $\frac{3}{2}.10^{-7}$;
1·4471580, $\frac{3}{4}.10^{-7}$; 1·6232493, $\frac{3}{4}.10^{-7}$; 1·7993406, $\frac{3}{4}.10^{-7}$;
2·6444386, 2.10^{-7}; 2·7024306, 3.10^{-7}.

30. ·3424227, $\bar{3}$·5606673, ·6575773.

33. 1·1760912, 1·0791812, $\bar{2}$·3467875.

35. 12599·21. **36.** 43822080 yds. **38.** 41.

39. ·0931432. **40.** ·9279516, 2·9279523, 4·9279540.

41. 5·2721589, $\bar{2}$·2721571, 1·2721513.

42. 2674·116, 2674·114, ·2674176.

43. 377·6067, ·003776077, 3·776088.

44. ·6244781, ·6245500. **45.** 1·0578505, 1·0576553.

46. 9·8048933, 9·8049583. **47.** 9·9691068, 9·9690952.

48. 897·946. **49.** $A=63°\ 49'\ 9''$, $B=26°\ 10'\ 51''$, $c=593{\cdot}601$.

50. $b=1613{\cdot}694$, $c=3227{\cdot}388$, $B=30°$. **51.** $b=134{\cdot}8163$.

52. $a=3149{\cdot}056$. **53.** $85°\ 54'\ 4''$. **54.** $80°\ 24'\ 21''$.

55. $102°\ 1'\ 29''$.

56. $A=48°\ 11'\ 28''$, $B=58°\ 24'\ 44''$, $C=73°\ 23'\ 53''$.
57. $B=118°\ 53'\ 33''$, $C=11°\ 6'\ 27''$.
58. $B=41°\ 25'\ 26''$, $C=18°\ 34'\ 34''$, $a=35{\cdot}341$.
59. $B=126°\ 52'\ 11''$, $C=36°\ 52'\ 11''$.
60. $A=70°\ 22'\ 35''$, $C=62°\ 24'\ 25''$, or $A=15°\ 11'\ 15''$, $C=117°\ 35'\ 35''$.
61. $C=37°\ 25'\ 45''$, $A=95°\ 21'\ 15''$, $a=325{\cdot}1741$.
62. $C=5°\ 11'\ 21''$, $A=51°\ 21'\ 39''$, $a=327{\cdot}5311$. **63.** $63°\ 26'\ 6''$.
64. 3185·379 ft. **65.** 343·9880 ft. **66.** 178·1434 ft.
67. 1000 ft. nearly.

Examples XII. pp. 242—244.

1. (1) AE. (2) AD. (3) BC. (4) CA. (5) CB. (6) O.

2. A and B would meet at $\dfrac{ad-bc}{a-b}$ miles from the town.

If (i) $a>b$ and $a/c>b/d$ or (ii) $a<b$ and $a/c<b/d$, before reaching the town.

If (i) $a>b$ and $a/c<b/d$ or (ii) $a<b$ and $a/c>b/d$ after having reached the town.

They would meet at $\dfrac{c-d}{a-b}$ hours from the given moment.

If (i) $c>d$ and $a>b$ or (ii) $c<d$ and $a<b$ after the given moment.

If (i) $c>d$ and $a<b$ or (ii) $c<d$ and $a>b$ before the given moment.

9. $\pm\sqrt{48}$, $\pm\dfrac{7}{\sqrt{48}}$. **10.** $-\dfrac{2\sqrt{2}}{3}$. **11.** $\dfrac{-1}{\sqrt{5}}$.

13. $\{6n+(-1)^n\}\,30°$. **14.** $(6n+1)\,30°$. **15.** $(12n\pm1)\,30°$.

16. $n\,180°+19°$. **17.** $(2n+1)\,45°$. **18.** $\{4n+(-1)^n\}\,45°$.

19. $(2n+1)\,45°$. **20.** $(4n+1)\,45°$. **21.** $(6n\pm1)\,30°$. **22.** $(6n\pm1)\,60°$.

23. $(2n+1)\,45°$ or $(6n\pm1)\,30°$. **24.** $(2n+1)\,45°$ or $(6n\pm1)\,30°$.

25. $(2n+1)\,45°$. **26.** $n\,180°$ or $(6n\pm1)\,60°$. **27.** $(6n\pm1)\,60°$.

28. $(4n+3)\,45°$. **29.** $(6n\pm1)\,60°$. **30.** $\theta=\dfrac{(2n+1)\,\pi}{2\,(p+q)}$.

31. $n\,360°$. **32.** $(2n+1)\,45°$.

33.

	sin.	cos.	tan.
150^0	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
250^g	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1
765^0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
-60^0	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
$-\frac{2\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
945^0	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1
108^0	$\frac{1}{4}\sqrt{(10+2\sqrt{5})}$	$-\frac{1}{4}(\sqrt{5}-1)$	$-\sqrt{(5+2\sqrt{5})}$
$-\frac{\pi}{8}$	$-\frac{1}{2}\sqrt{(2-\sqrt{2})}$	$\frac{1}{2}\sqrt{(2+\sqrt{2})}$	$-\sqrt{2}+1$
$2n\pi+\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$(2n+1)\pi-\frac{\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
$2n\pi+\frac{5\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$

38.

	$\sin A + \cos A$	$\sin A - \cos A$	$\tan A + \cot A$	$\tan A - \cot A$
0 to 45°	inc. from 1 to $\sqrt{2}$	inc. from -1 to 0	dec. from ∞ to 2	inc. from $-\infty$ to 0
45° to 90°	dec. from $\sqrt{2}$ to 1	inc. from 0 to 1	inc. from 2 to ∞	inc. from 0 to ∞
90° to 135°	dec. from 1 to 0	inc. from 1 to $\sqrt{2}$	inc. from $-\infty$ to -2	inc. from $-\infty$ to 0
135° to 180°	dec. from 0 to -1	dec. from $\sqrt{2}$ to 1	dec. from -2 to $-\infty$	inc. from 0 to ∞
180° to 225°	dec. from -1 to $-\sqrt{2}$	dec. from 1 to 0	dec. from ∞ to 2	inc. from $-\infty$ to 0
225° to 270°	inc. from $-\sqrt{2}$ to -1	dec. from 0 to -1	inc. from 2 to ∞	inc. from 0 to ∞
270° to 315°	inc. from -1 to 0	dec. from -1 to $-\sqrt{2}$	inc. from $-\infty$ to -2	inc. from $-\infty$ to 0
315° to 360°	inc. from 0 to 1	inc. from $-\sqrt{2}$ to -1	dec. from -2 to $-\infty$	inc. from 0 to ∞

	$\sin A$. $\cos A$	$\tan A$. $\sec A$	$\cos A$. $\cot A$	$\frac{\cos A + \sin A}{\cos A - \sin A}$
0 to 45°	inc. from 0 to $\frac{1}{2}$	inc. from 0 to $\sqrt{2}$	dec. from ∞ to $\frac{1}{\sqrt{2}}$	inc. from 1 to ∞
45° to 90°	dec. from $\frac{1}{2}$ to 0	inc. from $\sqrt{2}$ to ∞	dec. from $\frac{1}{\sqrt{2}}$ to 0	inc. from $-\infty$ to -1
90° to 135°	dec. from 0 to $-\frac{1}{2}$	dec. from ∞ to $\sqrt{2}$	inc. from 0 to $\frac{1}{\sqrt{2}}$	inc. from -1 to 0
135° to 180°	inc. from $-\frac{1}{2}$ to 0	dec. from $\sqrt{2}$ to 0	inc. from $\frac{1}{\sqrt{2}}$ to ∞	inc. from 0 to 1
180° to 225°	inc. from 0 to $\frac{1}{2}$	dec. from 0 to $-\sqrt{2}$	inc. from $-\infty$ to $-\frac{1}{\sqrt{2}}$	inc. from 1 to ∞
225° to 270°	dec. from $\frac{1}{2}$ to 0	dec. from $-\sqrt{2}$ to $-\infty$	inc. from $-\frac{1}{\sqrt{2}}$ to 0	inc. from $-\infty$ to -1
270° to 315°	dec. from 0 to $-\frac{1}{2}$	inc. from $-\infty$ to $-\sqrt{2}$	dec. from 0 to $-\frac{1}{\sqrt{2}}$	inc. from -1 to 0
315° to 360°	inc. from $-\frac{1}{2}$ to 0	inc. from $-\sqrt{2}$ to 0	dec. from $-\frac{1}{\sqrt{2}}$ to $-\infty$	inc. from 0 to 1

EXAMPLES XIV. pp. 277—280.

25. $1-8\cos^2 A+8\cos^4 A$, $8\cos^3 A-4\cos A$. **48.** $(3n\pm1)\,60°$.

49. $(2n\pm1)\,45°$. **50.** $\{4n+(-1)^n\}\,45°$ or $(2n+1)\,22\frac{1}{2}°$.

51. $(8n\pm1)\,45°$ or $(2n+1)\,22\frac{1}{2}°$.

52. $\{6n+(-1)^n\}\,30°$, $\{10n+(-1)^n\}\,18°$ or $\{10n-3(-1)^n\}\,18°$.

53. $(4n+1)\,30°$ or $(4n-1)\,45°$.

54. $n180°$, $n180°+67\frac{1}{2}°$, or $n180°-22\frac{1}{2}°$.

55. $(2n+1)\,90°$, $(10n\pm1)\,18°$ or $n180°\pm54°$.

56. $(2n+1)\,90°$ or $\{4n+(-1)^n\}\,15°$. **57.** $n\,.\,180°$ or $\{4n+(-1)^n\}15°$.

58. $n\,.\,90°$ or $(3n\pm1)\,120°$. **59.** $n\,.\,180°$ or $(6n\pm1)\,10°$.

60. $(2n+1)\,90°$, or $(3n\pm1)\,40°$. **61.** $(2n+1)\,90°$ or $(6n\pm1)\,20°$.

62. $n\,.\,180$ or $\frac{1}{2}\cos^{-1}\frac{\sqrt{3}-1}{2}$. **63.** $n\,.\,180°$. **64.** $(2n+1)\,15°$.

65. $(12n\pm1)\,30°$. **66.** $\frac{\pi}{4}+\sin^{-1}\frac{m}{\sqrt[6]{2}}$.

67. $(2n+1)\,22\frac{1}{2}°$ or $(4n+1)\,18°$. **68.** $\frac{1}{2}$. **69.** 0 or $\frac{1}{2}$.

70. $\tan^{-1}\frac{1}{2057}$.

EXAMPLES XV. pp. 299—302.

1. *a.* $\{3n+(-1)^n\}\,30°$. *b.* $(12n\pm1)\,15°$. *c.* $n\,.\,90°+37\frac{1}{2}°$.
d. $n\,.\,90°+7\frac{1}{2}°$. *e.* $(6n\pm1)\,20°$. *f.* $\{10n+(-1)^n\}6°$.
g. $n\,.\,60+(-1)^n45°$. *h.* $(3n-1)\,20°$. *k.* $n\,.\,120°$.
l. $n\,.\,60°$.

2. $\sin 9°=+\frac{1}{2}\sqrt{(1+\sin 18°)}-\frac{1}{2}\sqrt{(1-\sin 18°)}=\frac{1}{4}\sqrt{(3+\sqrt{5})}-\frac{1}{4}\sqrt{(5-\sqrt{5})}$.
$\cos 9°=+\frac{1}{2}\sqrt{(1+\sin 18°)}+\frac{1}{2}\sqrt{(1-\sin 18°)}=\frac{1}{4}\sqrt{(3+\sqrt{5})}+\frac{1}{4}\sqrt{(5-\sqrt{5})}$.
$\cot 9°=\operatorname{cosec}18°+\cot 18°=\sqrt{5}+1+\sqrt{(5+2\sqrt{5})}$.
$\tan 9°=\operatorname{cosec}18°-\cot 18°=\sqrt{5}+1-\sqrt{(5+2\sqrt{5})}$.
$\operatorname{cox}9°=\sqrt{(1+\cot^2 9°)}=\sqrt{(7+3\sqrt{5})}+\sqrt{(5+\sqrt{5})}$.
$\sec 9°=\sqrt{(1+\tan^2 9°)}=\sqrt{(7+3\sqrt{5})}-\sqrt{(5+\sqrt{5})}$.
$\sin 4\frac{1}{2}°=\sin(22\frac{1}{2}°-18°)=\frac{1}{8}\sqrt{(10+2\sqrt{5})}\sqrt{(2-\sqrt{2})}-\frac{1}{8}(\sqrt{5}-1)\sqrt{(2+\sqrt{2})}$.
$\cos 4\frac{1}{2}°=\cos(22\frac{1}{2}°-18°)=\frac{1}{8}\sqrt{(10+2\sqrt{5})}\sqrt{(2+\sqrt{2})}+\frac{1}{8}(\sqrt{5}-1)\sqrt{(2-\sqrt{2})}$.
$\cot 4\frac{1}{2}°=\operatorname{cox}9°+\cot 9°=\sqrt{(7+3\sqrt{5})}+\sqrt{(5+\sqrt{5})}+\sqrt{5}+1+\sqrt{(5+2\sqrt{5})}$.
$\tan 4\frac{1}{2}°=\operatorname{cox}9°-\cot 9°=\sqrt{(7+3\sqrt{5})}+\sqrt{(5+\sqrt{5})}-\sqrt{5}-1-\sqrt{(5+2\sqrt{5})}$.
$\sin 6°=\sin(60°-54°)=\frac{1}{8}\{\sqrt{(30+6\sqrt{5})}-\sqrt{5}-1\}$.
$\cos 6°=\cos(60°-54°)=\frac{1}{8}\{\sqrt{(10+2\sqrt{5})}+\sqrt{15}+\sqrt{3}\}$.
$\sin 7\frac{1}{2}°=\sin(22\frac{1}{2}°-15°)=\frac{1}{8}(\sqrt{6}+\sqrt{2})\sqrt{(2-\sqrt{2})}-\frac{1}{8}(\sqrt{6}-\sqrt{2})\sqrt{(2+\sqrt{2})}$.
$\cos 7\frac{1}{2}°=\cos(22\frac{1}{2}°-15°)=\frac{1}{8}(\sqrt{6}+\sqrt{2})\sqrt{(2+\sqrt{2})}+\frac{1}{8}(\sqrt{6}-\sqrt{2})\sqrt{(2-\sqrt{2})}$.
$\cot 7\frac{1}{2}°=\operatorname{cox}15°+\cot 15°=\sqrt{6}+\sqrt{2}+2+\sqrt{3}$.
$\tan 7\frac{1}{2}°=\operatorname{cox}15°-\cot 15°=\sqrt{6}+\sqrt{2}-2-\sqrt{3}$.
$\sin 3°=\sin(18°-15°)=\frac{1}{16}(\sqrt{6}+\sqrt{2})(\sqrt{5}-1)-\frac{1}{16}(\sqrt{6}-\sqrt{2})\sqrt{(10+2\sqrt{5})}$.
$\cos 3°=\cos(18°-15°)=\frac{1}{16}(\sqrt{6}+\sqrt{2})\sqrt{(10+2\sqrt{5})}+\frac{1}{16}(\sqrt{6}-\sqrt{2})(\sqrt{5}-1)$.

3. $2\cos A/2^n = \sqrt{[2+\sqrt{\{2+\sqrt{(2 \text{ to } n \text{ roots} + 2\cos A)}\}}]}$.

4.

A goes	$\sin\frac{1}{2}A + \cos\frac{1}{2}A$	$\sin\frac{1}{2}A - \cos\frac{1}{2}A$
0 to 90^0	inc. from 1 to $\sqrt{2}$	inc. from -1 to 0
90^0 to 180^0	dec. from $\sqrt{2}$ to 1	inc. from 0 to 1
180^0 to 270^0	dec. from 1 to 0	inc. from 1 to $\sqrt{2}$
270^0 to 360^0	dec. from 0 to -1	dec. from $\sqrt{2}$ to 1

27. $-41^\circ\ 36'\ 41''$. **32.** $(1-2\lambda)\pi$, $2\mu\pi$, $2(\lambda-\mu)\pi$.

33. $x = d\cos(s-\alpha)$, $y = d\cos(s-\beta)$, $z = d\cos(s-\gamma)$ if $s = \frac{1}{2}(\alpha+\beta+\gamma)$, where α, β, γ and d may have either sign throughout; and it is assumed that $\pm\alpha\pm\beta\pm\gamma$ is not equal to $2r\pi$.

EXAMPLES XVI. pp. 318, 319.

9. (*a*) $8x^3 = 4x^2 + 4x + 1$. (*b*) $64x^3 - 112x^2 + 56x = 7$.

EXAMPLES XVII. pp. 323, 324.

6. $\dfrac{\cos\{2\alpha+(n-1)\beta\}\sin n\beta}{2\sin\beta} + \dfrac{n}{2}$.

7. $\dfrac{3\sin\{\alpha+\frac{1}{2}(n-1)\beta\}\sin\frac{1}{2}n\beta}{4\sin\frac{1}{2}\beta} - \dfrac{\sin\{3\alpha+\frac{3}{2}(n-1)\beta\}\sin\frac{3}{2}n\beta}{4\sin\frac{3}{2}\beta}$.

8. $\dfrac{\cos\{4\alpha+2(n-1)\beta\}\sin 2n\beta}{8\sin 2\beta} + \dfrac{\cos\{2\alpha+(n-1)\beta\}\sin n\beta}{2\sin\beta} + \dfrac{3n}{8}$.

9. $\dfrac{\cos(2\theta+n\alpha)\sin n\alpha}{2\sin\alpha} + \dfrac{n}{2}\cos\alpha$.

10. $\dfrac{\cos(n+2)\theta\,.\sin n\theta}{2\sin\theta} + \dfrac{n}{2}\sin\theta$.

11. $\dfrac{2\sin n\alpha}{\cos(n+1)\alpha\,.\sin 2\alpha}$.

12. $\dfrac{\sin n\alpha}{\cos(n+1)\alpha\,.\sin 2\alpha}$.

13. $\dfrac{\sin 2\alpha}{2} - \dfrac{\sin 2^{n+1}\alpha}{2^{n+1}}$.

14. $\dfrac{3\sin\alpha}{4} - \dfrac{\sin 3^n\alpha}{3^{n-1}.4}$.

15. $\dfrac{3\cos\alpha}{4} + (-1)^{n-1}\dfrac{\cos 3^n\alpha}{3^{n-1}.4}$.

16. $\cos 2\theta - \cos 2^{n+1}\theta$.

17. $\frac{1}{2}\operatorname{cosec}^2\frac{1}{2}\theta - 2^{n-1}\operatorname{cosec}^2 2^{n-1}\theta$.

18. $\dfrac{\sec\alpha - \sec(2n+1)\alpha}{2\sin\alpha}$.

19. $2(\cot\alpha - \cot 2^n\alpha)$.

20. $\dfrac{\tan^2(n+1)\alpha - \tan^2\alpha}{\sin\alpha}$.

21. $\dfrac{\sin n\theta\,(3\cos n\theta\cot\theta-\sin n\theta)-3n\cos 2n\theta}{2\sin\theta}$.

22. $\dfrac{1}{8^{n-1}}\cot\dfrac{x}{2^{n-1}}\operatorname{cosec}^2\dfrac{x}{2^{n-1}}-8\cot 2x\,.\operatorname{cosec}^2 2x$.

23. $\dfrac{\sin\frac{1}{2}(3n+1)a\sin\frac{3}{2}na}{2\sin\frac{3}{2}a}+\dfrac{\sin\frac{1}{2}(n-1)a\sin\frac{1}{2}na}{2\sin\frac{1}{2}a}$.

24. $2\cos a-2\sin a\,.\cot 2^n a$.

25. $\frac{1}{8}[3^n\tan 3^n a-\tan a]$.

26. $\frac{1}{8}[\cot a-3^n\cot 3^n a]$.

27. $\dfrac{\sin a-x^n\sin(n+1)a+x^{n+1}\sin na}{1-2x\cos a+x^2}$.

28. $\dfrac{\cos a-x-x^n\cos(n+1)a+x^{n+1}\cos na}{1-2x\cos a+x^2}$.

29. $\dfrac{\sin a+x-(-x)^n\cos(n+1)(\frac{1}{2}\pi-a)+(-x)^{n+1}\cos n(\frac{1}{2}\pi-a)}{1+2x\sin a+x^2}$.

30. $\dfrac{\cos a-x^n\sin(n+1)(\frac{1}{2}\pi-a)+x^{n+1}\sin n(\frac{1}{2}\pi-a)}{1-2x\sin a+x^2}$.

31. $\dfrac{(n+1)\sin na-n\sin(n+1)a}{2(1-\cos a)}$.

32. $\dfrac{(n+1)\cos na-n\cos(n+1)a-1}{2(1-\cos a)}$.

33. $\frac{1}{2}\tan 2n\phi$.

34. $\tan 2n\phi-\tan\phi$.

Examples XIX. pp. 388—393.

11. $1,\ e^{-2},\ e^{-\frac{1}{2}\cdot\frac{a^2}{\beta^2}},\ e^{\frac{a^2}{2\beta^2}}$.

17. 1.

18. $\log a-\log b$.

19. $\frac{1}{4}$.

20. $2p^2/q^2$.

21. 1.

22. 1.

23. 0.

24. $\frac{1}{6\,0}$.

25. $\frac{1}{3\,6\,0}$.

26. $\frac{1}{2}m$.

27. $2(n^2-m^2)/p^2$.

28. -30.

29. 4.

30. -9.

31. 5^{-5}.

32. e.

33. e^3.

50. $\dfrac{x\sin ha}{1-2x\cosh a+x^2}$.

51. $\frac{1}{2}(e^{xe^a}+ex^{e^{-a}})-1$.

52. $\frac{1}{2}(e^{-e^\beta\cos a}+e^{-e^{-\beta}\cos a})$.

53. $\frac{1}{2}\log\dfrac{1+e^a}{1+e^{-a}}$.

54. $-\frac{1}{2}\log(1-2\cos\theta\,.\cosh\phi+\cos^2\theta)$.

55. $\frac{1}{2}\{1+\cos a(\cosh\beta+\sinh\beta)\}^{\frac{p}{q}}-\frac{1}{2}\{1+\cos a(\cosh\beta-\sinh\beta)\}^{\frac{p}{q}}$.

82. $\dfrac{2n(2n+1)x^2}{(2n+2)(2n+3)-x^2}$, $\dfrac{(2n-1)2nx^2}{(2n+1)(2n+2)-x^2}$.

Examples XX. pp. 413—415.

1. ·0020944, ·008799, ·01745, ·0381.

2. $\dfrac{2160}{\pi}$ in.

4. 1·1090966.

7. 1·4641147.

10. 8·4647449, 8·4647488.

11. 3° 14′ 28″ ·1307.

EXAMPLES XXI. pp. 465—471.

7. $1^{\frac{1}{4}} = +1$ or -1 or $+i$ or $-i$. $(-1)^{\frac{1}{4}} = \frac{1+i}{\sqrt{2}}$ or $\frac{-1-i}{\sqrt{2}}$ or $\frac{1-i}{\sqrt{2}}$ or $\frac{-1+i}{\sqrt{2}}$.

The above eight quantities are the values of $1^{\frac{1}{8}}$.

$(-1)^{\frac{1}{8}} = \pm\frac{1}{2}\sqrt{(2+\sqrt{2})} \pm \frac{1}{2}i\sqrt{(2-\sqrt{2})}$ or $\pm\frac{1}{2}\sqrt{(2-\sqrt{2})} \pm \frac{1}{2}i\sqrt{(2+\sqrt{2})}$.

13. $x^5+1 \equiv (x+1)(x^4-x^3+x^2-x+1) = 0$ gives $x = -1$ or

$x^2+2+x^{-2} = (x+x^{-1})+\frac{1}{4} = \frac{5}{4}$, $\therefore\ x+x^{-1} = \frac{1}{2}(\pm\sqrt{5}+1)$,

$\therefore\ x = \frac{1}{4}\{(\sqrt{5}+1) \pm i\sqrt{(10-2\sqrt{5})}\}$ or $\frac{1}{4}\{(-\sqrt{5}+1) \pm i\sqrt{(10+2\sqrt{5})}\}$.

Trigonometrically: $(-1)^{\frac{1}{5}} = \cos\frac{1}{5}(2r+1)\,180° + i\sin\frac{1}{5}(2r+1)\,180°$,

$= \cos 36° + i\sin 36°$, or $\cos 108° + i\sin 108°$, or $\cos 180° + i\sin 180°$, or $\cos 252° + i\sin 252°$, or $\cos 324° + i\sin 324°$.

14. $1+i = \sqrt{2}(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi)$. $\sqrt{3}+i = 2(\cos\frac{1}{6}\pi + i\sin\frac{1}{6}\pi)$.

$1-i\sqrt{3} = 2(\cos\frac{5}{3}\pi + i\sin\frac{5}{3}\pi)$. $-1-i = \sqrt{2}(\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi)$.

$\sqrt{2}+1+i = \sqrt{(4+2\sqrt{2})}(\cos\frac{1}{8}\pi + i\sin\frac{1}{8}\pi)$.

$2-\sqrt{3}+i = \sqrt{(8-4\sqrt{3})}(\cos\frac{5}{12}\pi + i\sin\frac{5}{12}\pi)$.

15. $1^{\frac{1}{6}} = \pm 1$ or $\frac{1}{2}(\pm 1 \pm i\sqrt{3})$. $(-1)^{\frac{1}{6}} = \pm i$ or $\frac{1}{2}(\pm\sqrt{3} \pm i)$.

$i^{\frac{1}{6}} = \pm\frac{1}{2}(\sqrt{2} - i\sqrt{2})$ or $\pm\frac{1}{2}\{(\sqrt{6}+\sqrt{2}) + i(\sqrt{6}-\sqrt{2})\}$

or $\pm\frac{1}{2}\{(\sqrt{6}-\sqrt{2}) + i(\sqrt{6}+\sqrt{2})\}$.

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